Existence for nonlinear parabolic PDEs

Idea: use linear theory + fixed point theorem.

Banach's fixed point thm

If (X, d) is a complete metric space, and $A : X \to X$ is a contraction (i.e. $\exists \gamma < 1$ st $d(Ax, Ay) \le \gamma d(x, y)$), then A has a unique fixed point, i.e. $\exists ! \, x^* \in X$ st. $Ax^* = x^*$.

Proof Select $x_0 \in X$. Then $x_k := A x_{k-1}$ is Cauchy, hence $x_k \to x^*$. $\boxed{/\!/}$

Ex: Reaction-diffusion equations

$$(*)\begin{cases} (\partial_t - \Delta) u = f(u) & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = g & \text{on } \Omega \times \{t=0\} \end{cases}$$

where $g \in H_0^1(\Omega)$, $f: \mathbb{R} \to \mathbb{R}$ Lipschitz,
in part. $|f(z)| \leq C(1+|z|)$.

Def $u \in L^2([0,T]; H_0^1(\Omega))$ with $u' \in L^2([0,T]; H^{-1}(\Omega))$
is a weak solution of $(*)$ if

(i) $\langle u'(t), v\rangle + B[u(t), v] = (f(u(t)), v) \quad \forall v \in H_0^1(\Omega)$
a.e. $t \in [0,T]$

(pairing H^{-1}, H_0^1) $\quad$ ($\int_\Omega \nabla u(t) \cdot \nabla v$) $\quad$ (L^2 inner product)

(ii) $u(0) = g$

Recall: $u \in L^2 H_0^1$, $u' \in L^2 H^{-1} \Rightarrow u \in C L^2$,
hence (ii) makes sense.

Thm There exists a unique weak solution of (*).

Proof We will apply Banach's FPT in the space

$$X := C([0,T]; L^2(\Omega))$$

with $\|u\| := \max_{0 \le t \le T} \|u(t)\|_{L^2(\Omega)}$.

Define A as follows:

Given $u \in X$, set $h(x,t) := f(u(x,t))$

$\Rightarrow h \in L^2L^2$ (in fact $L^\infty L^2$)

Hence we can solve

$$\begin{cases} (\partial_t - \Delta) w = h & \text{in } \Omega_T \\ w = 0 & \text{on } \partial\Omega \times [0,T] \\ w = g & \text{on } \Omega \times \{t=0\} \end{cases}$$

Namely, by linear existence theory,

$\exists!$ weak solution

$w \in L^2 H_0^1$ with $w' \in L^2 H^{-1}$ ($\Rightarrow w \in X$)

$$\begin{cases} \langle w', v \rangle + B[w,v] = (h, v) \quad \text{a.e. } t, \forall v \in H_0^1 \\ w(0) = g \end{cases}$$

Define $A : X \to X$ by $A[u] := w$.

<u>Claim:</u> If T is small enough, then

A is a contraction.

<u>Proof</u> Let $u, \tilde{u} \in X$. Set $w := A[u]$, $\tilde{w} := A[\tilde{u}]$.

Compute

$$\frac{1}{2}\frac{d}{dt}\|w - \tilde{w}\|_{L^2}^2 = \langle w' - \tilde{w}', w - \tilde{w} \rangle$$

$$= (\Delta w + h - \Delta \tilde{w} - \tilde{h}, w - \tilde{w})$$

$$= -\left(\nabla(w-\tilde{w}), \nabla(w-\tilde{w})\right) + \left(h-\tilde{h}, w-\tilde{w}\right).$$

Hence,

$$\frac{d}{dt}\|w-\tilde{w}\|_{L^2}^2 + 2\|w-\tilde{w}\|_{H_0^1}^2$$

$$= 2\left(w-\tilde{w}, h-\tilde{h}\right)$$

$$\leq \varepsilon\|w-\tilde{w}\|_{L^2}^2 + \frac{1}{\varepsilon}\|f(u)-f(\tilde{u})\|_{L^2}^2$$

$$\leq \varepsilon C_p\|w-\tilde{w}\|_{H_0^1}^2 + \frac{1}{\varepsilon}\|f(u)-f(\tilde{u})\|_{L^2}^2$$

Choose $\varepsilon = 1/C_p$, get:

$$\frac{d}{dt}\|w-\tilde{w}\|_{L^2}^2 \leq C\|u-\tilde{u}\|_{L^2}^2 .$$

Thus $$\|w(s)-\tilde{w}(s)\|_{L^2}^2 \leq C\int_0^s \|u(t)-\tilde{u}(t)\|_{L^2}^2\,dt$$

$$\leq CT\|u-\tilde{u}\|_X^2$$

Take sup over $s \in [0,T]$, get

$$\|w - \tilde{w}\|_X^2 \le CT \|u - \hat{u}\|_X^2$$

Hence

$$\|A[u] - A[\hat{u}]\|_X \le (CT)^{1/2} \|u - \hat{u}\|_X$$

and A is a contraction for $T < C^{-1}$ $\boxed{//}$

Now, let T_1 be st $(CT_1)^{1/2} = \frac{1}{2}$.

Banach FPT $\Rightarrow \exists!$ weak sol. on $[0, T_1]$.

Note that C is universal, i.e. independent of the solution.

Since $u(t) \in H_0^1$ for a.e. t,

we can repeat the argument to extend our sol. to $[T_1, 2T_1]$ etc until $[0,T]$.

uniqueness: $u, \tilde{u}$ two weak sol

$\Rightarrow w = u, \hat{w} = \tilde{u}$

$$\Rightarrow \quad \|u(s) - \tilde{u}(s)\|_{L^2}^2 \leq C \int_0^s \|u(t) - \tilde{u}(t)\|_{L^2}^2 \, dt$$

Gronwall $\Rightarrow u \equiv \tilde{u}$. $\square$