Maximum principle

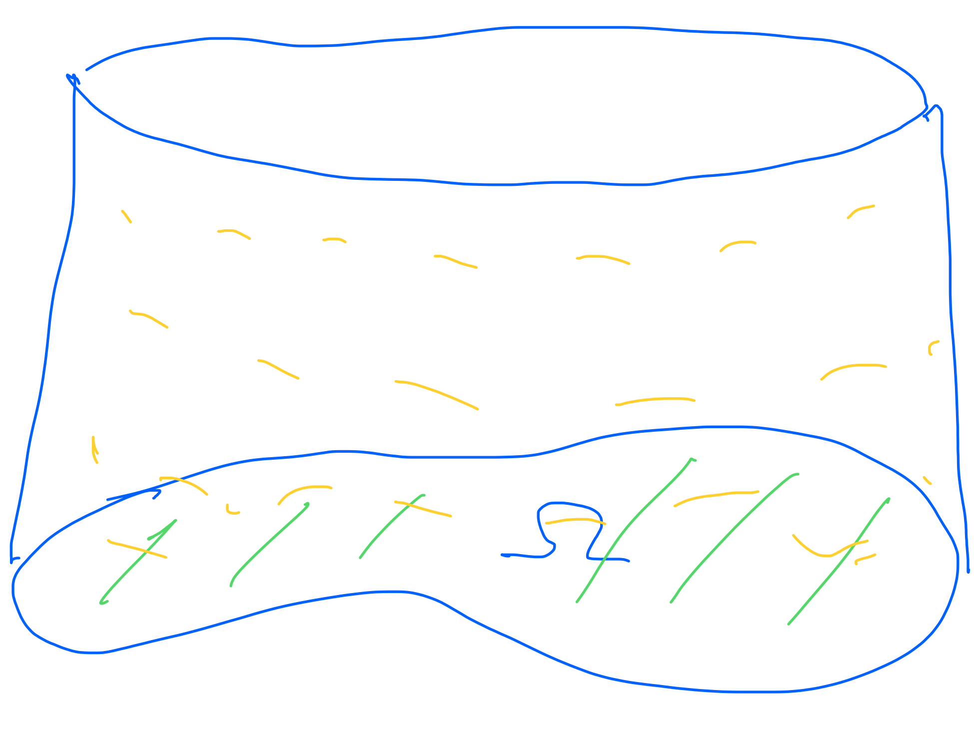
$$Lu = -\sum_{ij} a^{ij} u_{x_i x_j} + \sum_i b^i u_{x_i} + cu$$

(nondivergence form)

Assume $a^{ij}, b^i, c \in C(\overline{\Omega_T})$

$$\sum_{ij} a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, (x,t) \in \Omega_T.$$

Recall $\Omega_T = \Omega \times (0,T]$ is the parabolic cylinder. Its parabolic boundary is $\Gamma_T = \overline{\Omega_T} \setminus \Omega_T = \underbrace{\Omega \times \{t=0\}}_{\text{bottom}} \cup \underbrace{\partial\Omega \times [0,T]}_{\text{sides}}$



Thm (weak maximum principle)

Assume $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$

(cont. differentiable once in time & twice in space)

and suppose $c \equiv 0$.

(i) If $(\partial_t + L)u \leq 0$ in Ω_T, (subsolution)

then $\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$

(ii) If $(\partial_t + L)u \geq 0$ in Ω_T (supersolution)

then $\min_{\overline{\Omega_T}} u = \min_{\Gamma_T} u$

Proof (1) Assume first $(\partial_t + L)u < 0$ in Ω_T.

Suppose towards a contradiction

that $\exists\,(x_0, t_0) \in \Omega_T$ with

$$u(x_0, t_0) = \max_{\overline{\Omega_T}} u$$

(a) If $0 < t_0 < T$

$x_0 \in \Omega$

$$\Rightarrow \begin{cases} \partial_t u = 0 & \text{at } (x_0, t_0) \\ Lu \geq 0 & \text{at } (x_0, t_0) \end{cases}$$

$$\Rightarrow (\partial_t + L)u \geq 0 \text{ at } (x_0, t_0) \ \text{↯}.$$

(b) If $t_0 = T$, then

$\partial_t u \geq 0$ at (x_0, t_0)

and still $Lu \geq 0$ at (x_0, t_0)

$$\Rightarrow (\partial_t + L)u \geq 0 \text{ at } (x_0, t_0) \ \text{↯}.$$

(2) In the general case $(\partial_t + L)u \leq 0$ in Ω_T

Consider $u_\varepsilon(x,t) := u(x,t) - \varepsilon t$.

Then $(\partial_t + L)u_\varepsilon = \underbrace{(\partial_t + L)u}_{\leq 0} - \varepsilon < 0$ in Ω_T

By part (1) we have

$$\max_{\overline{\Omega_T}} u_\varepsilon = \max_{\Gamma_T} u_\varepsilon .$$

$$\varepsilon \to 0 \implies \max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u .$$

(3) For part (ii) apply part (i) to $-u$. $\boxtimes$

<u>Thm</u> (Weak max. principle for $c \geq 0$)

Assume $u \in C^2_1(\Omega_T) \cap C(\overline{\Omega_T})$ and $c \geq 0$

(i) If $(\partial_t + L)u \le 0$ in Ω_T

then $\max_{\overline{\Omega}_T} u \le \max_{\Gamma_T} u^+$.

(ii) If $(\partial_t + L)u \ge 0$ in Ω_T

then $\min_{\overline{\Omega}_T} u \ge - \max_{\Gamma_T} u^-$.

In particular, if $(\partial_t + L)u = 0$ in Ω_T,

then $\max_{\overline{\Omega}_T} |u| = \max_{\Gamma_T} |u|$.

Proof (1) Assume first $(\partial_t + L)u < 0$ in Ω_T

Suppose towards a contradiction that u attains a positive maximum at $(x_0, t_0) \in \Omega_T$.

Since $u(x_0, t_0) > 0$, $c \geq 0$

$\Rightarrow (\partial_t + L) u \geq 0$ at (x_0, t_0) ↯.

(2) In general $(\partial_t + L) \leq 0$ in Ω_T.

As before, consider $u_\varepsilon(x,t) := u(x,t) - \varepsilon t$

$\Rightarrow (\partial_t + L) u_\varepsilon < 0$ in Ω_T.

Furthermore, if u attains a positive max in Ω_T, then so does u_ε, provided ε is small enough.

$\rightsquigarrow$ contradiction as before. $\boxed{\text{Z}}$

$(\partial_t - \Delta) u = 0$

$\Rightarrow (\partial_t - \Delta) u^2 \leq 0$

Also $(\partial_t - \Delta) |\nabla u|^2 \leq 0$

want to prove: $t\,|\nabla u|^2 \leq C$

$$(\partial_t - \Delta)(t\,|\nabla u|^2) \leq \underbrace{|\nabla u|^2}_{\text{bad term.}}$$

$$(\partial_t - \Delta)\, u^2 \leq \underbrace{-2|\nabla u|^2}_{\text{good term}}$$

$$t\,|\nabla u|^2 + \alpha u^2$$

as long as $\alpha \geq \frac{1}{2}$ that's a subsolution.

Weak maximum principle on nonconnect domains

In general need some growth condition to get max. princ. / uniqueness.

Ex (Tychonoff)

Let $f(t) = \begin{cases} e^{-1/t^2} & t \neq 0 \\ 0 & t = 0 \end{cases}$

Then $u(x,t) := \sum_{n=0}^{\infty} f^{(n)}(t) \dfrac{x^{2n}}{(2n)!}$

is a nontrivial smooth solution of $\partial_t u = \partial_x^2 u$ with $u(x,0) \equiv 0$.

Thm If $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C^0(\mathbb{R}^n \times [0,T])$

is a solution of $\partial_t u = \Delta u \quad (x,t) \in \mathbb{R}^n \times [0,T]$

such that $u(x,t) \leq A e^{B|x|^2}$ on $\mathbb{R}^n \times [0,T]$

for some $A, B < \infty$,

then $\sup_{(x,t) \in \mathbb{R}^n \times [0,T]} u(x,t) = \sup_{x \in \mathbb{R}^n} u(x,0)$.

<u>Proof</u> If $\sup_{x\in\mathbb{R}^n} u(x,0) = \infty$ nothing to prove.

Assume now $D := \sup_{x\in\mathbb{R}^n} u(x,0) < \infty$.

Consider

$$v(x,t) := u(x,t) - \frac{\delta}{(T'+\varepsilon-t)^{n/2}} e^{\frac{|x|^2}{4(T'+\varepsilon-t)}}$$

Note that $(\partial_t - \Delta) v = 0$ on $\mathbb{R}^n \times (0,T')$

Choose $T' \leq T$ small enough s.t. $4BT' < 1$.

$$\Rightarrow \frac{1}{4(T'+\varepsilon-t)} \geq \frac{1}{4(T'+\varepsilon)} > B$$

for $t\in[0,T')$ and ε suff small.

growth assumption $\Rightarrow v(x,t) \to -\infty$ as $|x|\to\infty$ uniformly for $t\in[0,T')$.

Let $H(x,t) := v(x,t) - D - \delta t$. ($D$ large)

<u>Claim</u> $\sup_{x\in\mathbb{R}^n} H(x,t) < 0 \quad \forall t\in[0,T']$

<u>Proof of Claim</u>

If not, since $v \to -\infty$ at spatial infinity and since $H(x,0) < 0$

$\Rightarrow \exists$ first time $t_0 \in (0,T']$ & point $x_0 \in \mathbb{R}^n$ s.t. $H(x_0,t_0) = \max_{\mathbb{R}^n\times[0,t_0]} H = 0$.

$\Rightarrow \partial_t H \geq 0, \nabla H = 0, \Delta H \leq 0$ at (x_0,t_0)

$\Rightarrow 0 = (\partial_t - \Delta) v = (\partial_t - \Delta) H + \delta > 0$ ↯ $\boxed{\geq}$

Thus $$\sup_{(x,t)\in\mathbb{R}^n\times[0,T']} v(x,t) \leq \sup_{x\in\mathbb{R}^n} u(x,0) + \delta T'$$

$$\delta \to 0 \implies \sup_{(x,t)\in\mathbb{R}^n\times[0,T']} u(x,t) \leq \sup_{x\in\mathbb{R}^n} u(x,0)$$

repeat on $[T', \min\{2T', T\}]$, etc. $\square$