Parabolic equations

Motivation:

heat eqn $\begin{cases} \partial_t u = \Delta u \\ u|_{t=0} = u_0 \end{cases}$ $\leftarrow$ say bdd & cont.

$u = u(x,t),\ x \in \mathbb{R}^n,\ t \geq 0.$

$$\Rightarrow \quad u(x,t) = \int_{\mathbb{R}^n} \underbrace{\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}}_{\text{heat kernel}} u_0(y)\, d^n y$$

→ becomes immediately smooth:

$$\| D^k u_t \|_{L^\infty} \leq \frac{C_k}{t^{k/2}} \| u_0 \|_{L^\infty}$$

→ exists $\forall t \geq 0$ and is unique

→ for $t \to \infty$ converges to steady state $\Delta u_\infty = 0$. Namely $u_\infty = \text{const}$.

→ time scales like distance squared, i.e. if $u(x,t)$ solves the heat equ, then so does $u(\lambda x, \lambda^2 t)$.

nonlinear heat equ

$$\begin{cases} \partial_t u = \Delta u + u^2 \\ u|_{t=0} = u_0 \in C(\mathbb{R}^n) \end{cases}$$

Eg if $u_0 \geq c > 0$:

$$\frac{d}{dt} u_{min} \geq u_{min}^2 \Rightarrow u_{min} \geq \frac{c}{1-ct}$$

$$\Rightarrow u_{min} \to \infty \text{ in } t \leq 1/c.$$

general u_0:

Sometimes diffusion wins

sometimes reaction wins.

·) smooth solution exists on maximal time interval $[0,T)$.

·) weak solutions important to continue evolution beyond first singular time.

·) study singularities, eg size of singular set, blowup limits.

Geometric heat equs

·) $\partial_t u = \Delta_{g,h} u$ harmonic map heat flow for $u:(M,g) \to (N,h)$

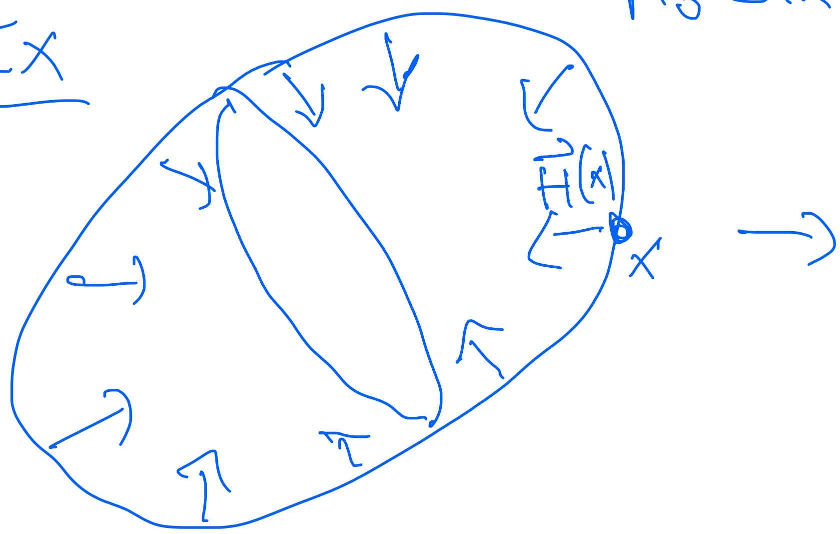
·) $\partial_t g = -2Rc(g)$ Ricci flow

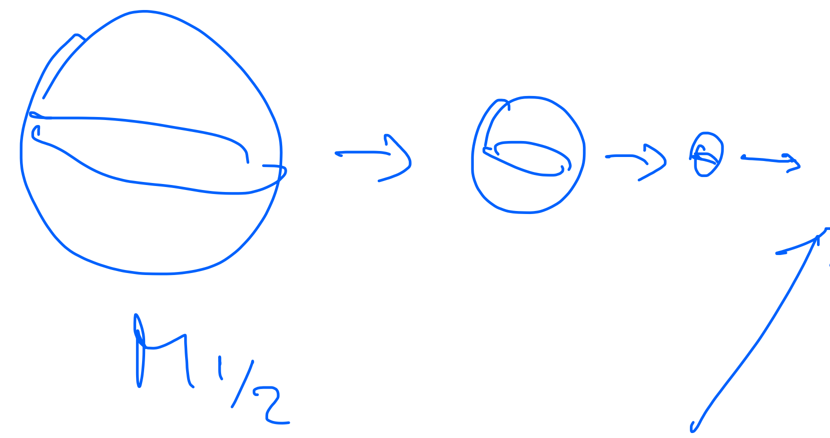
·) $\partial_t x = \vec{H}(x)$ mean curvature flow ("heat equ for surfaces")

Ex

$M_0 \subset \mathbb{R}^3$

" $\vec{H}(x) = \Delta_{M_t} x$ "

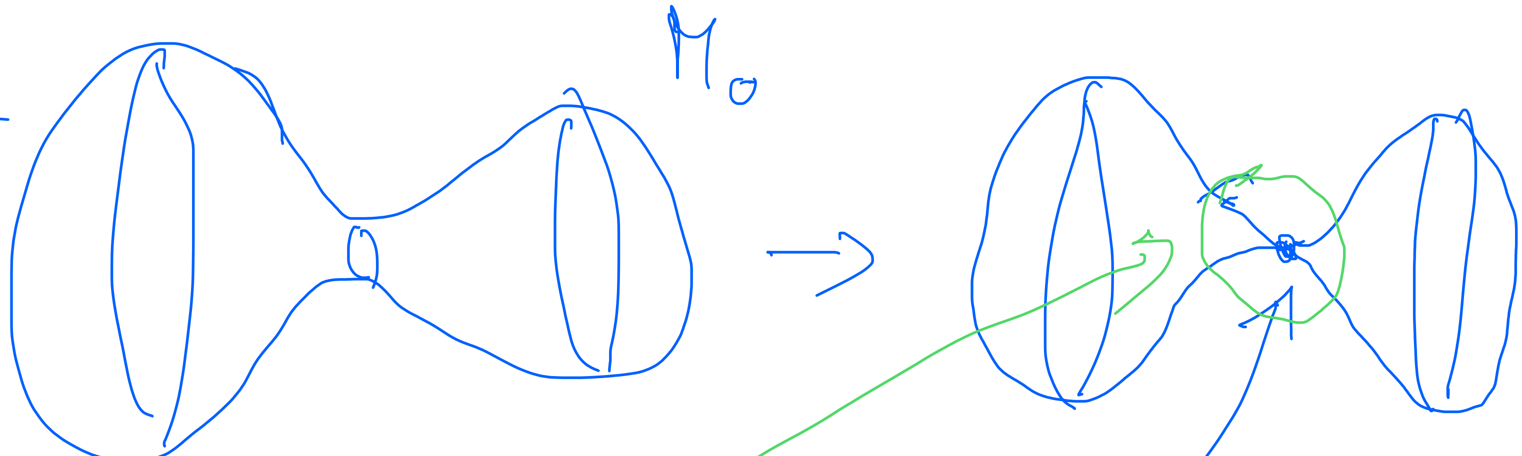




becomes extinct in a "round point"

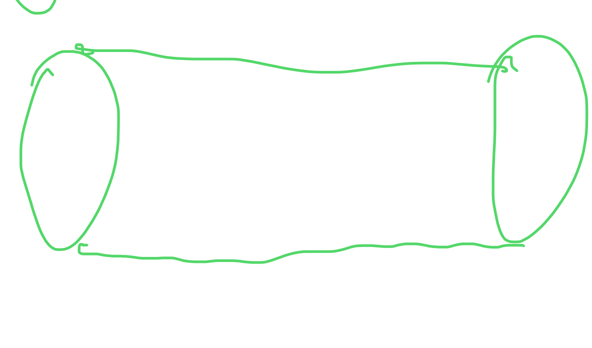
Ex

M_0



neckpinch singularity

blowup limit = round shrinking cylinder



Preliminaries: Banach space-valued functions

want to view $u(x,t)$, where $x \in \Omega \subset \mathbb{R}^n$, $t \in [0,T]$

as $u : [0,T] \longrightarrow X$
$t \longmapsto u(\cdot, t)$ eg $X = H_0^1(\Omega)$

$\rightarrow$ have to define spaces like $L^2([0,T]; H_0^1(\Omega))$

Measurability & Integration (Evans App E)

Def: $f : [0,T] \longrightarrow (X, \|.\|)$ is called

-) weakly measurable if $\forall L \in X^*$ the mapping $t \mapsto \langle L, f(t) \rangle$ is Lebesgue measurable.
-) strongly measurable if there exist simple functions $s_k : [0,T] \to X$ st
$s_k(t) \longrightarrow f(t)$ in X for a.e. $t \in [0,T]$.

Def: $s : [0,T] \to X$ is called simple,
if $s(t) = \sum_{i=1}^{m} \chi_{E_i}(t)\, u_i$,

where $E_i \subseteq [0,t]$ Lebesgue measurable
and $u_i \in X$.

Thm (Pettis) If X is seperable, then

weakly measurable $\Leftrightarrow$ strongly measurable.

(general X: strongly measurable $\Leftrightarrow$ weakly measurable & almost seperably valued)

Def: A strongly measurable function $f : [0,T] \to X$ is integrable if $\exists\, s_k$ simple st

$$\int_0^T \| s_k(t) - f(t) \| \, dt \to 0 .$$

In this case we define

$$\int_0^T f(t)\,dt := \lim_{k\to\infty} \int_0^T s_k(t)\,dt$$

Here, for $s(t) = \sum_{i=1}^m \chi_{E_i}(t)\, u_i$

we set $\int_0^T s(t)\,dt := \sum_{i=1}^m |E_i|\, u_i$.

Thm (Bochner) A strongly measurable function $f : [0,T] \to X$ is integrable $\iff t \mapsto \|f(t)\|$ is integrable.

In this case:

•) $\left\| \int_0^T f(t)\,dt \right\| \leq \int_0^T \|f(t)\|\,dt$

•) $\left\langle L, \int_0^T f(t)\,dt \right\rangle = \int_0^T \langle L, f(t)\rangle\,dt$

$\forall L \in X^*$.

Banach space valued function spaces (Evans 5.9)

Def The space $L^p([0,T];X)$ consist of all strongly measurable $u:[0,T]\to X$ st

$$\|u\|_{L^p([0,T];X)} := \left(\int_0^T \|u(t)\|^p dt\right)^{1/p} < \infty \quad (1\le p<\infty)$$

respectively $\|u\|_{L^\infty} := \operatorname*{esssup}_{0\le t\le T} \|u(t)\| < \infty \quad (p=\infty)$

Def $C([0,T];X)$ consists of all continuous $u:[0,T]\to X$ with

$$\|u\|_{C([0,T];X)} := \max_{0\le t\le T} \|u(t)\| < \infty.$$

Def Let $u,v\in L^1([0,T];X)$. We say $u'=v$ weakly if

$$\int_0^T \phi'(t)\,u(t)\,dt = -\int_0^T \phi(t)\,v(t)\,dt$$

$$\forall \phi\in C_c^\infty((0,T))$$

Def $W^{1,p}([0,T];X) := \{ u \in L^p([0,T];X) \mid u' \in L^p \}$

$$\|u\|_{W^{1,p}} := \left(\int_0^T \left(\|u(t)\|^p + \|u'(t)\|^p \right) dt \right)^{1/p}$$

respectively $\operatorname{ess\,sup}_{0 \le t \le T} \|u(t)\| + \|u'(t)\|$ for $p = \infty$.

Thm Let $u \in W^{1,p}([0,T];X)$. Then

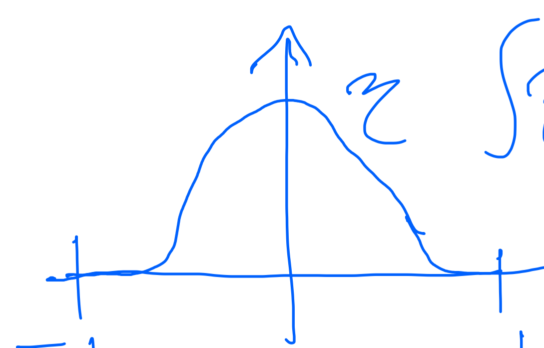
(i) $u \in C([0,T];X)$

(ii) $u(t) = u(s) + \int_s^t u'(\tau)\, d\tau$

(iii) $\max_{t \in [0,T]} \|u(t)\| \le C_T \|u\|_{W^{1,p}([0,T];X)}$

Proof extend u by 0 on $\mathbb{R} \setminus [0,T]$.

Set $u^\varepsilon = \eta_\varepsilon * u$



$\int \eta = 1$

where $\eta_\varepsilon(t) = \frac{1}{\varepsilon} \eta\left(\frac{t}{\varepsilon}\right)$.

As $\varepsilon \to 0$ we have

$$\begin{cases} u^\varepsilon \to u & \text{in } L^p([0,T];X) \\ (u^\varepsilon)' \to u' & \text{in } L^p_{loc}([0,T];X) \end{cases}$$

Fix $0 < s < t < T$. Compute

$$u^\varepsilon(t) = u^\varepsilon(s) + \int_s^t (u^\varepsilon)'(\tau)\, d\tau$$

$$\Rightarrow u(t) = u(s) + \int_s^t u'(\tau)\, d\tau$$

for a.e. $0 < s < t < T$.

Since $t \mapsto \int_0^t u'(\tau)\, d\tau$ is continuous this yields (i) & (ii).

Finally, (ii) $\Rightarrow$ (iii) $\boxed{//}$

Thm Suppose $u \in L^2([0,T]; H^1_0(\Omega))$
and $u' \in L^2([0,T]; H^{-1}(\Omega))$

Then: (i) $u \in C([0,T]; L^2(\Omega))$

(ii) $t \mapsto \|u(t)\|^2_{L^2(\Omega)}$ abs. cont,

$$\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} = 2\langle u'(t), u(t)\rangle$$

for a.e. $t \in [0,T]$.

(iii) $$\max_{0 \le t \le T} \|u(t)\|_{L^2(\Omega)} \le C_T\left(\|u\|_{L^2([0,T]; H^1_0(\Omega))} + \|u'\|_{L^2([0,T]; H^{-1}(\Omega))}\right)$$

Proof $u^\varepsilon := \eta_\varepsilon * u$ as before.

$$\frac{d}{dt}\|u^\varepsilon(t) - u^\delta(t)\|^2_{L^2(\Omega)} =$$

$$= 2\left\langle u^{\varepsilon\prime}(t) - u^{\delta\prime}(t),\, u^{\varepsilon}(t) - u^{\delta}(t)\right\rangle_{L^2(\Omega)}$$

$$\Rightarrow \|u^{\varepsilon}(t) - u^{\delta}(t)\|^2_{L^2(\Omega)} = \|u^{\varepsilon}(s) - u^{\delta}(s)\|^2_{L^2(\Omega)}$$

$$+ 2\int_s^t \left\langle u^{\varepsilon\prime}(\tau) - u^{\delta\prime}(\tau),\, u^{\varepsilon}(\tau) - u^{\delta}(\tau)\right\rangle d\tau$$

Fix $s \in (0,T)$ with $u^{\varepsilon}(s) \to u(s)$ in $L^2(\Omega)$.

Then:

$$\limsup_{\varepsilon,\delta\to 0} \sup_{t\in[0,T]} \|u^{\varepsilon}(t) - u^{\delta}(t)\|^2_{L^2(\Omega)}$$

$$\leq \limsup_{\varepsilon,\delta\to 0} \int_0^T \Big(\|u^{\varepsilon\prime}(\tau) - u^{\delta\prime}(\tau)\|^2_{H^{-1}(\Omega)} + \|u^{\varepsilon}(\tau) - u^{\delta}(\tau)\|^2_{H_0^1(\Omega)} \Big)\, d\tau$$

$$= 0 \quad \Rightarrow \text{(i)}.$$

Similarly,

$$\|u^\varepsilon(t)\|^2_{L^2(\Omega)} = \|u^\varepsilon(s)\|^2_{L^2(\Omega)} + 2\int_s^t \langle u^{\varepsilon\prime}(\tau), u^\varepsilon(\tau)\rangle d\tau$$

$$\Rightarrow \|u(t)\|^2_{L^2(\Omega)} = \|u(s)\|^2_{L^2(\Omega)} + 2\int_s^t \langle u'(\tau), u(\tau)\rangle d\tau$$

$$\Rightarrow \text{(ii)} \underset{\text{integrate}}{\Longrightarrow} \text{(iii)} \qquad \boxtimes$$

Thm Suppose $u \in L^2([0,T]; H^{m+2}(\Omega))$

$u' \in L^2([0,T]; H^m(\Omega))$

Then: (i) $u \in C([0,T]; H^{m+1}(\Omega))$

(ii) $\max\limits_{0\le t\le T} \|u(t)\|_{H^{m+1}(\Omega)} \le C(T,\Omega,m)\Big(\|u\|_{L^2([0,T],H^{m+2}(\Omega))}$

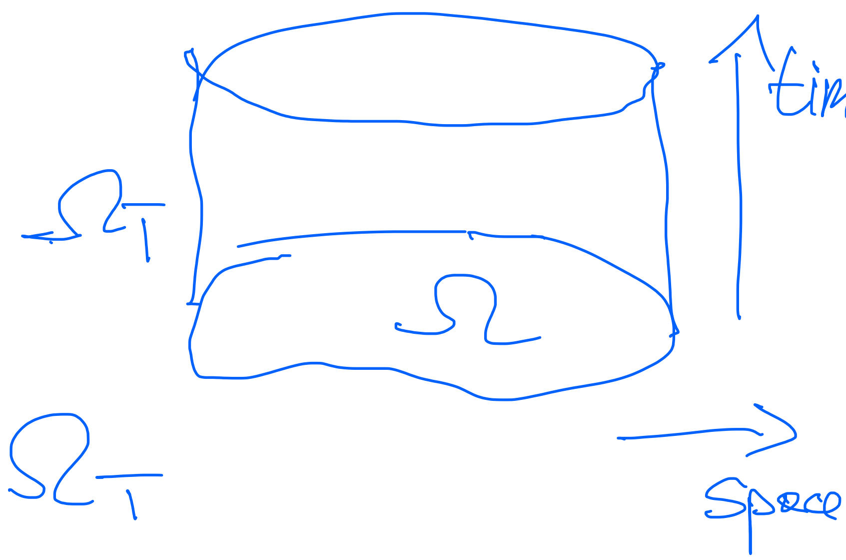
$+ \|u'\|_{L^2([0,T],H^m(\Omega))}\Big)$

Proof: similar $\boxtimes$

Linear 2^{nd} order parabolic eqns (Evans 7.1)

Setup $\Omega \subset \mathbb{R}^n$ smooth domain

$\Omega_T := \Omega \times (0, T]$ parabolic cylinder



$$(*) \begin{cases} u_t + Lu = f & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega \times [0, T] \\ u = g & \text{on } \Omega \times \{t = 0\} \end{cases}$$

$f : \Omega_T \to \mathbb{R}$, $g : \Omega \to \mathbb{R}$ given

$u : \overline{\Omega_T} \to \mathbb{R}$, $u = u(x,t)$ the unknown.

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x,t)\, u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^{i}(x,t)\, u_{x_i} + c(x,t)\, u$$

Assume $\exists\, \theta > 0$ st

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\, \xi_i \xi_j \geq \theta |\xi|^2 \quad \forall (x,t) \in \Omega_T$$
$$\forall \xi \in \mathbb{R}^n$$

i.e. $\partial_t + L$ is (uniformly) parabolic

(physical meaning of L:
diffusion + transport + creation)

Assume
$$\begin{cases} a^{ij},\, b^{i},\, c \in L^\infty(\Omega_T) \\ f \in L^2(\Omega_T) \\ g \in L^2(\Omega) \end{cases}$$

$$\partial_t u + Lu = f \quad \Big| \quad \text{multiply by } v \in H_0^1(\Omega) \text{ and integrate}$$

$$\Rightarrow \int_\Omega u' v + \underbrace{\int_\Omega \sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_i b^i u_{x_i} v + c u v}_{=: B[u,v;t]} = \int f v$$

__Def__ We say that a function $u \in L^2([0,T]; H_0^1(\Omega))$ with $u' \in L^2([0,T]; H^{-1}(\Omega))$ is a weak solution of $(*)$ if

(i) $\langle u', v \rangle + B[u, v; t] = (f, v)$ $\forall v \in H_0^1(\Omega)$ a.e. $t \in [0,T]$

(ii) $u(0) = g$

__Rmk__ (ii) makes sense since $u \in C([0,T]; L^2(\Omega))$ by prior thm.