De Giorgi-Nash-Moser estimates

$$Lu = D_i(a^{ij}D_j u) + b^i D_i u + cu$$

$$\|a,b,c\|_{L^\infty} \leq \Lambda, \quad a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2.$$

Thm (see eg GT 8.24)

Suppose $Lu = f \in L^q(B_2)$, $q > n/2$.

Then

$$\|u\|_{C^\alpha(B_1)} \leq C\left(\|u\|_{L^2(B_2)} + \|f\|_{L^q(B_2)}\right)$$

where $\alpha = \alpha(n,\lambda,\Lambda) > 0$, $C = C(n,\lambda,\Lambda,q) < \infty$.

Proof (Sketch)

I $L^2 \rightsquigarrow L^\infty$

$$-D_i(a_{ij}D_ju) = f \in L^3 \text{ on } T^4 = \mathbb{R}^4/\mathbb{Z}^4$$

Sobolev: $\|v\|_{L^4} \le C\|v\|_{H^1} \quad (n=4)$

1st step:

$$\lambda\int|Du|^2 \le \int D_iu\, a_{ij}\, D_ju$$

$$= \int fu \le \|f\|_{L^2}^2 + \|u\|_{L^2}^2$$

$$\Rightarrow \|u\|_{L^4} \le C_1\left(\|f\|_{L^2} + \|u\|_{L^2}\right).$$

2nd step:

$$\frac{3}{4}\lambda\int|Du^2|^2 \le \frac{3}{4}\int D_iu^2\, a_{ij}\, D_ju^2 \qquad \left(\frac{3}{4}D_iu^2 D_ju^2 = D_iu^3 D_ju\right)$$

$$= -\int u^3 D_i a_{ij} D_j u = \int f u^3$$

$$\Rightarrow \|u\|_{L^8} \le C_1C_2\left(\|f\|_{L^3} + \|u\|_{L^2}\right)$$

3rd step:

$$\frac{7}{8}\lambda\int|Du^4|^2 \le \ldots \int f u^7$$

$$\Rightarrow \|u\|_{L^{16}} \le C_1 C_2 C_3 \left(\|f\|_{L^3} + \|u\|_{L^2}\right).$$

- localize on balls of $r_k = 1 + 2^{-k}$.
- iterate & check that $\prod_k C_k < \infty$

(essentially since $\sum_k \frac{k^p}{2^k} < \infty$)

$$\Rightarrow \|u\|_{L^\infty(B_1)} \le C\left(\|f\|_{L^2} + \|u\|_{L^2}\right).$$

(HW2, Q2: boarderline case

$-D_i(a_{ij} D_j u) \le u^2 = c \cdot u$, $c = u \in L^2$

show iteration works, provided $\|u\|_{L^2} \le \varepsilon$)

II Harnack ineq / Oscillation est

- If $Lu = 0$, $u \geq 0$ in B_1

  then $\sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u$.

- Suppose $Lu = 0$ in B_2

  $M := \sup_{B_1} u$, $m := \inf_{B_1} u$

  Harnack for $M-u \Rightarrow M - u(x) \leq C \inf_{B_{1/2}} (M-u)$

  Harnack for $u-m \Rightarrow u(x) - m \leq C \inf_{B_{1/2}} (u-m)$

  $\forall x \in B_{1/2}$

$$\Rightarrow \operatorname{osc}_{B_1} u = M - m \leq C\left(\operatorname{osc}_{B_1} u - \operatorname{osc}_{B_{1/2}} u\right)$$

$$\Rightarrow \operatorname{osc}_{B_{1/2}} u \leq \gamma \operatorname{osc}_{B_1} u, \quad \gamma = 1 - C^{-1} < 1.$$

iterate $\Rightarrow \operatorname{osc}_{B_{2^{-k}}} u \le \gamma^k \operatorname{osc}_{B_1} u \Rightarrow$ Hölder $\boxed{/}$

<u>Interpolation Lemma</u> (Ehrling)

Assume $X \underset{\text{cpt}}{\subset\subset} Y \underset{\text{cont}}{\subseteq} Z$ Banach

Then $\forall \varepsilon > 0\ \exists C_\varepsilon < \infty$:

$$\|x\|_Y \le \varepsilon \|x\|_X + C_\varepsilon \|x\|_Z$$

(<u>Ex</u> $C^{2,\alpha} \subset\subset C^2 \subseteq L^\infty$;

$$\|u\|_{C^2} \le \varepsilon \|u\|_{C^{2,\alpha}} + C_\varepsilon \|u\|_{L^\infty} \Big)$$

<u>Proof</u> If not, $\exists \varepsilon > 0, x_k \in X$:

$$\varepsilon \|x_k\|_X + k \|x_k\|_Z < \|x_k\|_Y = 1$$

$\Rightarrow x_k \xrightarrow{\|\cdot\|_Y} y \in Y$ for a subsequence.

$$\Rightarrow \begin{cases} \|y\|_Y = 1 \Rightarrow y \neq 0 \\ \|y\|_Z = 0 \Rightarrow y = 0 \end{cases} \lightning \quad \boxed{//}$$

Cor The norms

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\sum_{|\gamma| \leq k} \int_\Omega |D^\gamma u|^p dx \Big)^{1/p}$$

and $\|u\| = \|u\|_{L^p(\Omega)} + \sum_{|\gamma| = k} \|D^\gamma u\|_{L^p(\Omega)}$

are equivalent.

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reminder: please submit HW1 today (crowdmark).