Schauder estimates

Setting: linear 2nd order elliptic operator
(non-divergence form)

$$Lu(x) = a^{ij}(x) D^2_{ij} u(x) + b^i(x) D_i u(x) + c(x) u(x)$$

$$\|a^{ij}\|_{C^\alpha(\Omega)}, \|b^i\|_{C^\alpha(\Omega)}, \|c\|_{C^\alpha(\Omega)} \leq \Lambda$$

$$a^{ij}(x)\xi_i\xi_j \geq \lambda |\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n$$

recall:

$$\|u\|_{C^{k,\alpha}(\Omega)} = \sum_{|\gamma| \leq k} \|D^\gamma u\|_{L^\infty(\Omega)} + \sum_{|\gamma| = k} |D^\gamma u|_\alpha,$$

where $$|f|_\alpha = \sup_{\substack{x,y\in\Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

<u>Main thm</u> (Interior Schauder est)

$$\|u\|_{C^{2,\alpha}(\Omega')} \le C\left(\|Lu\|_{C^{\alpha}(\Omega)} + \|u\|_{L^{\infty}(\Omega)}\right)$$

where $C = C(n, \alpha, \lambda, \Lambda, \Omega, \Omega') < \infty$.

<u>Rmks</u> i) enough to prove it for

$$\Omega = B_2(0), \quad \Omega' = B_1(0).$$

ii) Historically, first proved by estimating $u = \Gamma * f$ for Newtonian kernel

$$\Gamma(x) = \begin{cases} c_n |x|^{2-n} & (n \neq 2) \\ \frac{1}{2\pi} \log|x| & (n = 2) \end{cases}$$

see eg. GT: Chap 4 & 6.

We will instead give a more modern proof via blowup (due to L. Simon).

The key towards proving the Main Thm is:

<u>Thm</u> (Fundamental Schauder est)

$\exists\, C = C(n,\alpha) < \infty$ s.t.

$$\boxed{|D^2 u|_\alpha \leq C\,|\Delta u|_\alpha}$$

$\forall u \in C^{2,\alpha}(\mathbb{R}^n)$.

To prove this we need:

<u>Lemma</u> (Liouville lemma)

If $u : \mathbb{R}^n \to \mathbb{R}$ is a harmonic fn with $\sup_{B_r(0)} |u| \leq C r^{3-\varepsilon}$ $\forall r$, then u is a quadratic polynomial.

Proof $\Delta u = 0 \Rightarrow$

$$|D^\gamma u(x_0)| \leq \frac{C_\gamma}{r^{|\gamma|+n}} \|u\|_{L^1(B(x_0,r))}$$

See eg. Evans p. 29.

$$|\gamma| \geq 3 \Rightarrow \frac{C_\gamma}{r^{|\gamma|+n}} \|u\|_{L^1(B(x_0,r))} \xrightarrow{r\to\infty} 0$$

$\Rightarrow D^\gamma u(x_0) = 0$ whenever $|\gamma| \geq 3$.

$\Rightarrow u$ quadratic (harmonic) polynomial $\square$

Proof of Fund. Schauder est:

Suppose towards a contradiction the assertion does not hold.

$\Rightarrow \exists u_\ell \in C^{2,\alpha}(\mathbb{R}^n)$ st

$$|D^2 u_\ell|_\alpha > \ell\, |\Delta u_\ell|_\alpha$$

After replacing u_ℓ by $\lambda_\ell u_\ell$, where $\lambda_\ell = |D^2 u_\ell|_\alpha^{-1}$, we can assume

$$|D^2 u_\ell|_\alpha = 1, \quad |\Delta u_\ell|_\alpha < \ell^{-1}.$$

$\Rightarrow$ $\exists\, i,j,k \in \{1,\cdots,n\}$ st for ∞-many ℓ $\exists\, x_\ell \in \mathbb{R}^n$ and $h_\ell > 0$

st. $$\frac{|D^2_{ij} u_\ell(x_\ell + h_\ell e_k) - D^2_{ij} u_\ell(x_\ell)|}{h_\ell^\alpha} \geq \frac{1}{2n^3}.$$

Shift x_ℓ to origin, rescale by h_ℓ, i.e. consider

$$\tilde{u}_\ell(x) = h_\ell^{-2-\alpha} u_\ell(x_\ell + h_\ell x)$$

$$\Rightarrow \begin{cases} |D^2 \tilde{u}_\ell|_\alpha = 1, \quad |\Delta \tilde{u}_\ell|_\alpha < \ell^{-1} \\ |D^2_{ij} \tilde{u}_\ell(e_k) - D^2_{ij} \tilde{u}_\ell(0)| \geq \frac{1}{2n^3}. \end{cases}$$

After adding a suitable 2^{nd} order polynomial, we can assume that

$$\tilde{u}_\ell(0)=0,\ D\tilde{u}_\ell(0)=0,\ D^2\tilde{u}_\ell(0)=0$$

$$|D^2\tilde{u}_\ell|_\alpha = 1 \Rightarrow \exists\, \tilde{u}_{\ell'} \to u \text{ in } C^2_{loc}$$

Our limit u satisfies:

$$\begin{cases} u(0)=0,\ Du(0)=0,\ D^2u(0)=0 \\ |D^2u|_\alpha \le 1,\ \Delta u = 0,\ D^2_{ij}u(e_k)\neq 0. \end{cases}$$

Liouville lemma $\Rightarrow D^2u = \text{const}$ $\lightning$ $\boxed{\text{QED}}$

Proof of Main Thm

Fund. Schauder est + linear coord change

$\Rightarrow \exists C_1 = C_1(\alpha, n, \lambda) < \infty$ st if

$A = (a^{ij})$ is a pos. def. symm. matrix with eigenvalues between λ & λ^{-1}, then

$$|D^2 v|_\alpha \le C_1 \Big| \sum_{i,j=1}^n a^{ij} D^2_{ij} v \Big|_\alpha$$

$\forall v \in C^{2,\alpha}(\mathbb{R}^n)$.

Now let L be as in statement of Main Thm. Given $x_0 \in B_1(0)$ and given $v \in C^{2,\alpha}(B_\rho(x_0))$, $\rho < 1$, "freeze the coefficients a^{ij}", namely write

$$a^{ij}(x_0) D^2_{ij} v = Lv - (a^{ij} - a^{ij}(x_0)) D^2_{ij} v - b^i D_i v - cv.$$

Using $|fg|_\alpha \le \|f\|_{L^\infty} |g|_\alpha + |f|_\alpha \|g\|_{L^\infty}$

and α-Hölder continuity of coeffs

$$\Rightarrow \left| \left(a^{ij} - a^{ij}(x_0)\right) D_{ij}^2 v \right|_\alpha$$

$$\le \Lambda \rho^\alpha |D^2 v|_\alpha + \Lambda \|D^2 v\|_{L^\infty(B_\rho)}$$

Choosing ρ small enough s.t. $C_1 \Lambda \rho^\alpha \le 1/2$,

we get:

$$|D^2 v|_\alpha \le C_2 \left(|Lv|_\alpha + \|v\|_{C^2(B_\rho(x_0))} \right)$$

where $C_2 = C_2(n, \alpha, \lambda, \Lambda) < \infty$.

Applying this for $v = \xi u$, where ξ is a suitable cutoff fn, with various centers x_0, we get:

$$\|u\|_{C^{2,\alpha}(B_1)} \leq C_3 \left(\|Lu\|_{C^{\alpha}(B_2)} + \|u\|_{C^2(B_2)} \right)$$

for some $C_3 = C_3(n, \alpha, \lambda, \Lambda) < \infty$.

Finally, using the interpolation ineq,

$$\|v\|_{C^2(B_1)} \leq \varepsilon \|v\|_{C^{2,\alpha}(B_1)} + C_\varepsilon \|v\|_{L^\infty(B_1)}$$

we can replace $\|u\|_{C^2(B_2)}$ by $\|u\|_{L^\infty(B_2)}$.

(see lecture notes for details) $\boxed{}$