Useful things to remember from PDE I:

Sobolev & Hölder spaces (Evans §5)

$C^{k,\alpha}(\Omega)$ "$k+\alpha$ times differentiable"

$W^{k,p}(\Omega)$ "$k-\frac{n}{p}$ times differentiable"

$$\text{engineering-dim}\left(\left(\int_\Omega |D^k u|^p d^n x\right)^{1/p}\right) = \text{length}^{\frac{n}{p}-k}$$

- If $k-\frac{n}{p} \geq l+\alpha$, then

  $W^{k,p} \subseteq C^{l,\alpha}$ (Morrey,

  and "$>$" $\Rightarrow$ "$\subset\subset$" Arzela-Ascoli)

- If $k-\frac{n}{p} \geq l-\frac{n}{q}$ and $k \geq l$,

  then $W^{k,p} \subseteq W^{l,q}$ and "$>$" $\Rightarrow$ "$\subset\subset$"

  (Gagliardo-Nirenberg-Sobolev, Rellich-Kondrachov)

Boarderline case

$n=2$: $W^{1,2} \hookrightarrow L^p \quad \forall p<\infty$,

but $W^{1,2} \not\hookrightarrow L^\infty$

Ex $\quad u(x) = \log(1-\log|x|)$, $\Omega = B_1 \subseteq \mathbb{R}^2$.

(Can look up sharper results e.g. Moser-Trudinger inequality)

Hilbert's 19th problem

$$E[u] = \int_\Omega \left(L(Du) - fu \right) dx$$

$f \in C^\infty(\bar\Omega)$

$L \in C^\infty(\bar\Omega \times \mathbb{R}^n)$

uniformly convex

$u|_{\partial\Omega} = g \in C^\infty(\partial\Omega)$

$\downarrow$ direct method of Calc of Var

unique minimizer $u \in H^1$

$\downarrow$ energy estimates

$u \in H^2$

$\downarrow$ deGiorgi Nash Moser iteration

$u \in C^{1,\alpha}$

$\downarrow$ Schauder estimates

$u \in C^\infty$

Schauder est (next lecture)

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = f \quad \text{elliptic PDE}$$

$$f \in C^{k,\alpha} \implies u \in C^{k+2,\alpha}$$
$$a_{ij} \in C^{k,\alpha}$$

Apply in our problem:

$$a_{ij} = L_{p_i p_j}(Du)$$

$$u \in C^{1,\alpha} \implies a_{ij} \in C^{0,\alpha}$$
$$\implies u \in C^{2,\alpha}$$
$$\implies a_{ij} \in C^{1,\alpha}$$
$$\implies u \in C^{3,\alpha}$$
$$\vdots$$

(bootstrap) $\boxed{3}$

$H^2 \rightsquigarrow C^{1,\alpha}$ assume $f \in W^{1,q}$, $q > n/2$

(EL) $\displaystyle\int_\Omega \sum_i L_{p_i}(Du)\, v_{x_i}\, dx = \int_\Omega f v\, dx$

Let $w \in C_c^\infty$, fix k. Choose $v = w_{x_k}$

$$\Rightarrow \int_\Omega \sum_{i,j} L_{p_i p_j}(Du)\, u_{x_j x_k}\, w_{x_i}\, dx = \int_\Omega f_{x_k} w\, dx$$

$\forall w \in H_0^1$

Let $\tilde{u} := u_{x_k}$, $a_{ij} = L_{p_i p_j}(Du)$

$$\Rightarrow \int_\Omega \sum_{i,j} a_{ij}\, \tilde{u}_{x_j}\, w_{x_i}\, dx = \int_\Omega f_{x_k} w\, dx$$

$\Rightarrow \tilde{u} \in H^1$ is a weak solution

of $\displaystyle -\sum_{i,j} (a_{ij}\, \tilde{u}_{x_j})_{x_i} = f_{x_k}$

deGiorgi Nash Moser (see later)

$\Rightarrow \tilde{u} \in C^{\alpha}_{loc}$ with

$$\|\tilde{u}\|_{C^{\alpha}(\Omega')} \leq C\left(\|\tilde{u}\|_{L^2(\Omega)} + \|f_{x_k}\|_{L^q(\Omega)}\right)$$

$\Rightarrow u \in C^{1,\alpha}_{loc}$

Rmk: Similar boundary estimates.