$$E[u] = \int_\Omega L(Du(x), u(x), x)\,dx \quad , \quad L \text{ smooth.}$$

$$E : \mathcal{F} \to \mathbb{R} \cup \{+\infty\} \qquad 1 < q < \infty$$

$$\mathcal{F} = \{ u \in W^{1,q}(\Omega) \mid \mathrm{Tr}(u) = g \}$$

$g \in \mathrm{Im}(\mathrm{Tr})$ given bdry cond.

<u>last lecture:</u> L coercive & convex in p $\Longrightarrow$

$$\exists u \in \mathcal{F} : \quad E[u] = \inf_{w \in \mathcal{F}} E[w].$$

<u>Q:</u> Equation for minimizer?

growth conditions (*):

$$\begin{cases} |L(p,z,x)| \le C(|p|^q + |z|^q + 1) \\ |D_p L| + |D_z L| \le C(|p|^{q-1} + |z|^{q-1} + 1) \end{cases}$$

Thm (weak sol of EL equns)

Assume (*). If $u \in \mathcal{F}$ satisfies $E[u] = \inf_{w \in \mathcal{F}} E[w]$, then u is a weak solution of the EL equns, i.e.

$$\int_\Omega \left(\sum_{i=1}^n L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x) v \right) dx = 0$$

$$\forall v \in W_0^{1,q}(\Omega).$$

Proof Fix $v \in W_0^{1,q}(\Omega)$. Set

$$e(\tau) := E[u + \tau v].$$

$$\Rightarrow \frac{e(\tau) - e(0)}{\tau} = \int_\Omega \underbrace{\frac{L(Du + \tau Dv, u + \tau v, x) - L(Du, u, x)}{\tau}}_{=: L^\tau(x)} dx$$

Clearly $L^\tau(x) \xrightarrow{\tau \to 0} \sum_{i=1}^n L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x) v$ for a.e. x.

To justify taking the limit under the integral compute

$$L^\tau(x) = \frac{1}{\tau}\int_0^\tau \frac{d}{ds} L(Du+sDv, u+sv, x)\,ds$$

$$= \frac{1}{\tau}\int_0^\tau \left(\sum_{i=1}^n L_{p_i}(Du+sDv, u+sv, x)\, v_{x_i} + L_z(Du+sDv, u+sv, x)\, v \right) ds$$

$$(*) \Rightarrow |L^\tau(x)| \le C\left(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1 \right) \in L^1(\Omega).$$

dominated convergence thm $\Rightarrow$

$$0 = \varphi'(0) = \int_\Omega \sum_{i=1}^n L_{p_i}(Du, u, x)\, v_{x_i} + L_z(Du, u, x)\, v \, dx$$

(7)

Rmks: (i) Sometimes there are also nonminimizing solutions of the EL eqns.

(ii) If $(p,z) \mapsto L(p,z,x)$ is convex $\forall x \in \Omega$ then every weak solution is a minimizer.

Regularity

General Q: $u \in \mathcal{F} \subseteq W^{1,q}(\Omega)$

minimizer / weak sol of EL.

How regular is u?

Answer depends on L

Exer: $\operatorname{div}(|\nabla u|^{q-2}\nabla u) = f$

$q \neq 2$ degenerate elliptic eqn

↑ at points where $|\nabla u| = 0$.

$u \in C^{1,\alpha}$ optimal regularity.

Hilbert's 19th problem: Are solutions of regular problems in the calculus of variations always analytic?

regular problem:

$\Omega \subseteq \mathbb{R}^2$ smooth bounded domain

$g: \partial\Omega \to \mathbb{R}$ smooth bdry cond.

$u \in \mathcal{Y}$ minimizer of $E[w] = \int_\Omega L(Dw, w, x)\,dx$

$L(p, z, x)$ analytic, strictly convex in p.

Answer: Yes!

Solution has 3 steps:

$$W^{1,2} \rightsquigarrow W^{2,2} \rightsquigarrow C^{1,\alpha} \rightsquigarrow C^\infty, C^\omega$$

energy estimates — De Giorgi Nash Moser iteration — Schauder estimates

recall: for $n=2$: $W^{2,2} \hookrightarrow C^\beta = C^{0,\beta}$ $\forall\, \beta < 1$

remember Sobolev embedding thms:

$W^{k,p}(\Omega)$ ($\Omega \subseteq \mathbb{R}^n$) "$k - \frac{n}{p}$ times differentiable"

Hardest step: $C^{0,99} \hookrightarrow C^{1,0,01}$.

Focus now on:

$$E[u] = \int_\Omega (L(Du) - uf)\,dx \qquad q=2$$

here $f \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ smooth bdd domain.

Assume $\exists\ \theta > 0$, $\Theta < \infty$:

$$(*) \quad \theta|\xi|^2 \le \sum_{i,j=1}^n L_{p_i p_j}(p)\xi_i\xi_j \le \Theta|\xi|^2.$$

$\Rightarrow \exists!$ minimizer $u \in \mathcal{Y}$ and u satisfies the EL eqns

$$\int_\Omega \sum_{i=1}^n L_{p_i}(Du) v_{x_i}\,dx = \int_\Omega fv\,dx \quad (EL)$$

$\forall\ v \in W_0^{1,2}(\Omega)$.

1. Energy estimates

Toy example: $\Delta u = f$, $f \in L^2$, $u \in C_c^\infty(\Omega)$.

$$\int_\Omega |\nabla^2 u|^2 = \int (\Delta u)^2 = \int f^2$$

integration by parts

$$\Rightarrow \|u\|_{W^{2,2}} \leq C \|f\|_{L^2}.$$

additional complications in general:

a) need interior & bdry estimates
(↑ cutoff fn.)

b) need difference quotients.

Thm (energy estimates)

(i) Let $u \in W^{1,2}(\Omega)$ be a weak sol. of (EL). Assume (*).

Then $u \in W^{2,2}_{loc}(\Omega)$. More precisely, $\forall\, \Omega' \subset\subset \Omega$

$\exists\, C = C(\Omega', \Omega, \theta, \textcircled{H}, \|f\|_{L^2(\Omega)}) < \infty$

s.t. $\|u\|_{W^{2,2}(\Omega')} \leq C$.

(ii) If in addition $u \in W^{1,2}_0(\Omega)$

then $\|u\|_{W^{2,2}(\Omega)} \leq C \|f\|_{L^2(\Omega)}$.

Proof $f \in L^2(\Omega)$, $u \in H^1(\Omega)$

(EL) $\int_\Omega \sum_{i=1}^n L_{p_i}(Du)\, v_{x_i}\, dx = \int_\Omega f v\, dx \quad \forall v \in H^1_0(\Omega)$

(C) $\theta \leq L_{p_i p_j} \leq \textcircled{H}$

(i) Interior energy est

Fix $\Omega' \subset\subset \Omega$. Want to estimate $\|u\|_{H^2(\Omega')}$

Choose $\xi \in C_c^\infty(\Omega)$ with $\xi \equiv 1$ on Ω'.

Plug $v := -D_k^{-h}(\xi^2 D_k^h u)$ into (EL)

where $D_k^h u(x) = \dfrac{u(x+he_k)-u(x)}{h}$.

Get: $\displaystyle\sum_{i=1}^n \int_\Omega D_k^h\big(L_{p_i}(Du)\big)\big(\xi^2 D_k^h u\big)_{x_i}\,dx$

$$= -\int_\Omega D_k^{-h}\big(\xi^2 D_k^h u\big)\,dx$$

Now $D_k^h L_{p_i}(Du(x)) = \dfrac{L_{p_i}(Du(x+he_k)) - L_{p_i}(Du(x))}{h}$

$$= \frac{1}{h}\int_0^1 \frac{d}{ds} L_{p_i}\big(sDu(x+he_k)+(1-s)Du(x)\big)\,ds$$

$$= \sum_{j=1}^n a^{ij,h}(x)\, D_k^h u_{x_j}(x),$$

where $a^{ij,h}(x) = \displaystyle\int_0^1 L_{p_i p_j}\big(sDu(x+he_k)+(1-s)Du(x)\big)\,ds$

$$\Rightarrow \sum_{i,j=1}^{n} \int_\Omega \xi^2 a^{ij,h} D_k^h u_{x_j} D_k^h u_{x_i}\,dx + \sum_{i,j=1}^{n} \int_\Omega a^{ij,h} D_k^h u_{x_j} D_k^h (2\xi\xi_{x_i})$$

$$\geq_{(C)} \theta \int_\Omega \xi^2 |D_k^h Du|^2 = -\int_\Omega f D_k^{-h}(\xi^2 D_k^h u)$$

$$\leq 2 \text{(H)} \int_\Omega \xi\, |D_k^h Du|\, |D_k^h u|$$

$$\leq \frac{\theta}{4} \int_\Omega \xi^2 |D_k^h Du|^2 dx + C \int_\Omega |D_k^h u|^2$$

$$\Rightarrow \leq \frac{\theta}{4} \int_\Omega \xi^2 |D_k^h Du|^2 + C \int_\Omega (f^2 + |Du|^2).$$

$$\Rightarrow \frac{\theta}{2} \int_\Omega \xi^2 |D_k^h Du|^2 \leq C \int_\Omega (f^2 + |Du|^2)$$

$\xi \equiv 1$ on $\Omega' \Rightarrow u \in H^2_{loc}(\Omega)$,

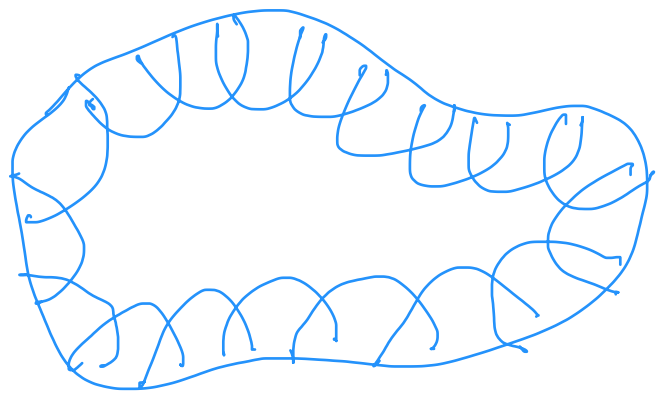
Ω' arbitrary $\quad \|u\|_{H^2(\Omega')} \leq C(\Omega, \Omega', \theta, \Theta)\left(\|f\|_{L^2(\Omega)} + \|u\|_{H^1}\right)$

∎

(ii) boundary / global energy est

$u \in H_0^1(\Omega)$

Claim: $\|u\|_{H^2(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\right)$



enough to prove it for halfball

$\xi \equiv 1$ on $B_{1/2}$, $\xi \equiv 0$ near

for $k = 1, \dots, n-1$ let $v = -D_k^{-h}\left(\xi^2 D_k^h u\right)$

argue as above

$$\Rightarrow \sum_{\substack{k+\ell < 2n \\ k,\ell = 1}}^{n} \|u_{x_k x_\ell}\|_{L^2} \le C\left(\|f\|_{L^2} + \|u\|_{H^1}\right)$$

estimate $u_{x_n x_n}$:

$$u \in H^2_{loc}(\Omega) \Rightarrow -\sum_{i,j=1}^{n} L_{p_i p_j}(Du)\, u_{x_i x_j} = f \text{ a.e.}$$

$$\Rightarrow \theta |u_{x_n x_n}| \le |L_{p_n p_n} u_{x_n x_n}|$$

$$\le \Big| \sum_{i+j \neq 2n} L_{p_i p_j}(Du)\, u_{x_i x_j} \Big| + |f|$$

$\Rightarrow$ Claim.

Finally, test with $v = u$ in EL

$$\Rightarrow \|u\|_{H^2(\Omega)} \le C \|f\|_{L^2(\Omega)}$$ □