Existence of minimizers

$$E[u] = \int_\Omega L(Du(x), u(x), x)\,dx$$

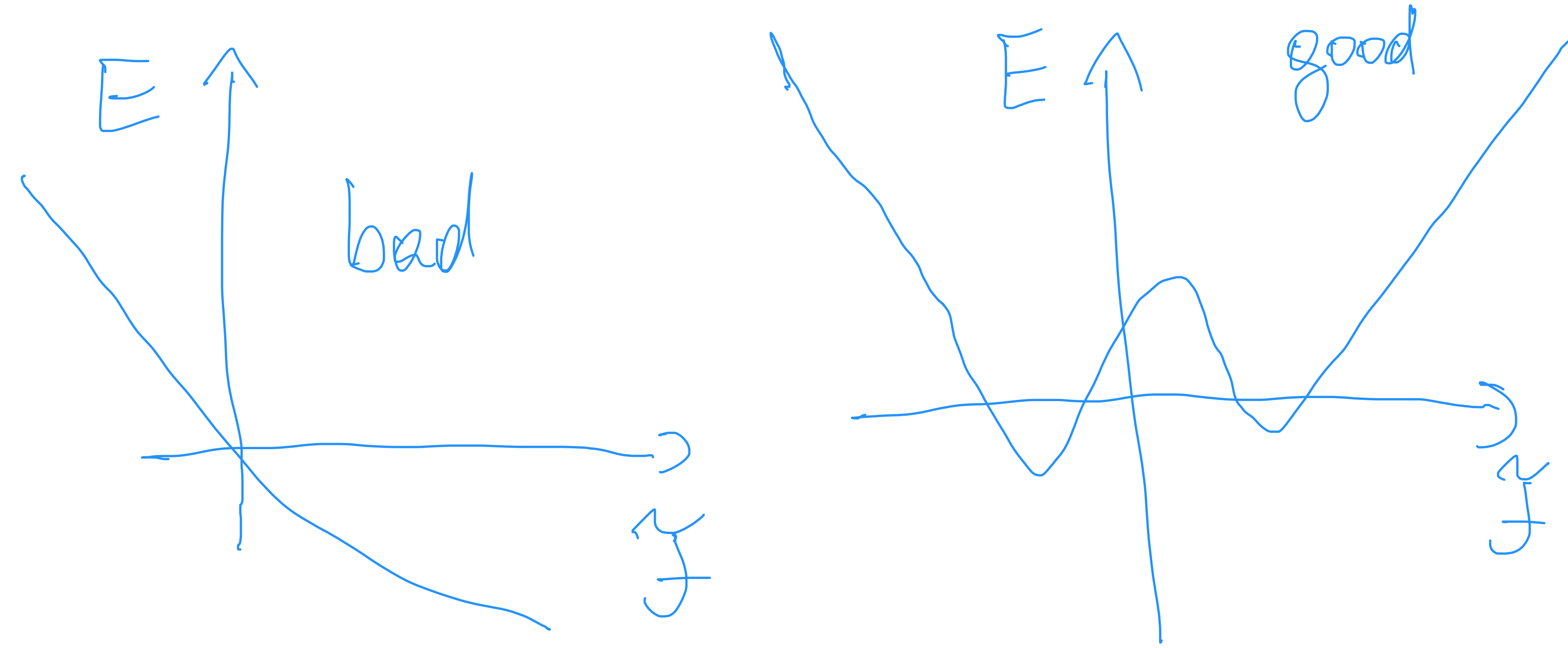
$$u : \Omega \to \mathbb{R}, \quad u|_{\partial\Omega} = g$$

$$E : \mathcal{F} \to \mathbb{R} \qquad \text{fix } 1 < q < \infty.$$

e.g. $\mathcal{F} = \{ u \in W^{1,q}(\Omega) \mid \underbrace{u|_{\partial\Omega}}_{} = g \}$

(recall trace thm from PDE I)

Q: Under what conditions can we find a minimizer?



<u>Def</u> E is called coercive (a)

if $E[u] \to \infty$ as $\|Du\|_{L^q} \to \infty$.

<u>Ex</u> If L satisfies the growth condition

$$L(p,z,x) \geq \alpha |p|^q - \beta$$

for some $\alpha > 0$, $\beta \in \mathbb{R}$, then

$$E[u] \geq \alpha \|Du\|_{L^q}^q - \beta|\Omega|,$$

and thus E is coercive.

(b) lower semicontinuity

recall: $\|u_k\|_{W^{1,q}(\Omega)} \le C \Rightarrow$

$u_{k_j} \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$

but in general $E[u_{k_j}] \not\to E[u]$.

<u>Def</u> E is <u>weakly lowersemicontinuous</u> on $W^{1,q}(\Omega)$, provided (l.s.c.)

$$E[u] \le \liminf_{k\to\infty} E[u_k]$$

whenever $u_k \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$.

<u>Thm 1</u> (weak lower semicontinuity)

Assume L is bounded below and

$p \mapsto L(p,z,x)$ is convex $\forall z\in\mathbb{R}, x\in\Omega$

Then $E[u] = \int_\Omega L(Du,u,x)\,dx$ is weakly l.s.c. on $W^{1,q}(\Omega)$

Rmk Last time: min (or finite index)

$$\Rightarrow \sum_{i,j=1}^{n} L_{p_i p_j}(Du, u, x)\xi_i \xi_j \geq 0$$

$$\forall \xi \in \mathbb{R}^n, \ \forall x \in \Omega.$$

Proof of Thm1 please read Evans p 446-447 $\square$

(c) $\mathcal{Y} \neq \emptyset$ also needed!

Thm 2 (Existence of minimizers)

Suppose $(W, \|\cdot\|)$ is a reflexive Banach space,

$\emptyset \neq \mathcal{Y} \subset W$ weakly closed subset,

$E : \mathcal{Y} \to \mathbb{R} \cup \{+\infty\}$ coercive & weakly l.s.c., i.e. (a) $E[u] \to \infty$ as $\|u\| \to \infty$

(b) $\forall u \in \mathcal{Y}$, $u_n \in \mathcal{Y}$ s.t. $u_n \rightharpoonup u$ weakly in W

we have $E[u] \le \liminf_{n\to\infty} E[u_n]$

Then E is bounded below and

$\exists\, u_{min} \in \mathcal{Y}$ s.t.

$$\boxed{E[u_{min}] = \inf_{u\in\mathcal{Y}} E[u]}$$

Proof Let $m := \inf_{u \in \mathcal{Y}} E[u] \in [-\infty, \infty]$.

Let $u_k \in \mathcal{Y}$ s.t. $E[u_k] \to m$.

coercivity $\Rightarrow \|u_k\| \le C$

W reflexive $\underset{\text{Eberlein-Smulian}}{\Rightarrow}$ $u_{k_j} \rightharpoonup u$ weakly in W.

$\mathcal{Y}$ weakly closed $\Rightarrow u \in \mathcal{Y}$

weak l.s.c $\Rightarrow$

$$E[u] \leq \liminf_{j\to\infty} E[u_{k_j}] = m.$$

$$\Rightarrow \quad E[u] = m > -\infty \qquad \boxed{//}$$

<u>Application:</u>

$W = W^{1,q}(\Omega)$ reflexive ✓

$Tr = |_{\partial\Omega} : W^{1,q}(\Omega) \to L^q(\partial\Omega)$

trace operator

$$\mathcal{F} = \{ u \in W^{1,q}(\Omega) \mid Tr(u) = g \}$$

•) $\mathcal{F}$ weakly closed:

If $u_k \in \mathcal{F}$, $u_k \rightharpoonup u$ in $W^{1,q}$

$\Rightarrow u_k - u_1 \in W_0^{1,q}(\Omega) \subset W^{1,q}(\Omega)$

weakly closed by Mazur's thm.

$\Rightarrow u - u_1 \in W_0^{1,q}(\Omega)$

$\Rightarrow Tr(u) = Tr(u_1) = g \Rightarrow u \in \mathcal{Y}$.

•) assume $L(p,z,x) \geq \alpha |p|^q - \beta$
$p \mapsto L(p,z,x)$ convex

$\Rightarrow E$ is coercive & weakly lsc.

•) $\mathcal{Y} \neq \emptyset$ for $g \in Im(Tr)$,
e.g. g Lipschitz.

<u>Rmk</u> (uniqueness)

In general nonunique.

Unique if E strictly convex,

e.g. $L = L(p,x)$, $\sum_{i,j} L_{p_i p_j} \xi_i \xi_j \geq \theta |\xi|^2$