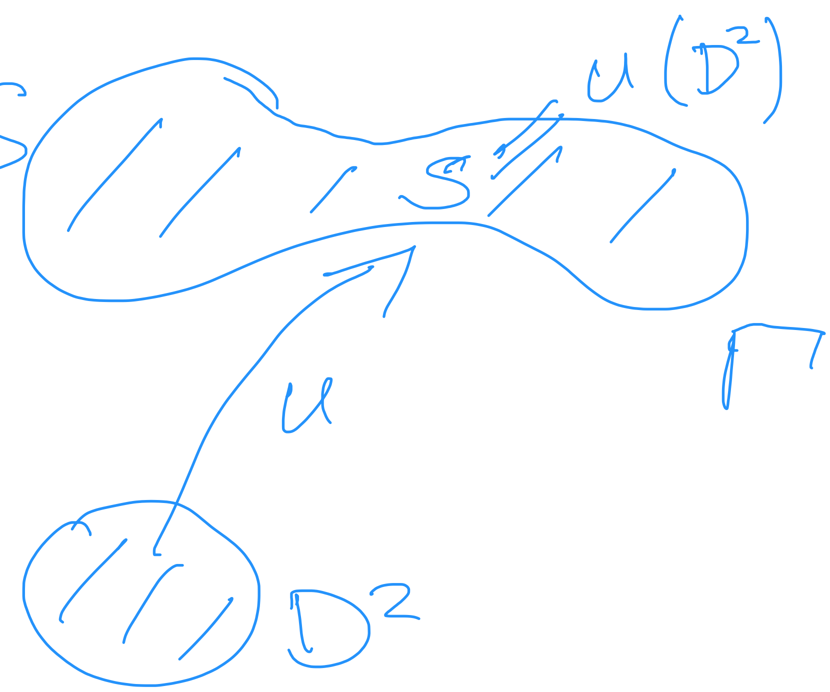
I. Elliptic PDEs & Calculus of Variations

Motivation: Plateau problem

$\Gamma \subset \mathbb{R}^3$ simple closed curve

Q: Is there a surface S with minimal area with $\partial S = \Gamma$?



Naive approach:

$u : D^2 \longrightarrow \mathbb{R}^3$ parametrization

$$A(u) = \int_D \sqrt{u_x^2 u_y^2 - (u_x \cdot u_y)^2}\, dx\, dy$$

$$A := \inf_{u \in C^\infty(D^2, \mathbb{R}^3) \text{ with } u(\partial D^2) = \Gamma} A(u).$$

Let u_i be a minimizing sequence,
i.e. $A(u_i) \to A$.

Q: Can we find a convergent subsequence?

Problem: lack of compactness

i) $A(u)$ does not control higher derivatives of u.
$\rightsquigarrow$ consider $W^{1,2}(D^2, \mathbb{R}^3)$ instead of $C^\infty(D^2, \mathbb{R}^3)$.

ii) $A(u)$ is invariant under an ∞-dim group of symmetries.
Namely, for any diffeomorphism $\varphi: D^2 \to D^2$ we have $A(u) = A(u \circ \varphi)$.

Idea (Douglas, Rado):

Instead of the area functional $A(u)$ consider the energy functional

$$E(u) = \frac{1}{2}\int_D |\nabla u|^2 dxdy = \frac{1}{2}\int_D (u_x^2 + u_y^2) dxdy$$

Note that $A(u) \leq E(u)$

with equality $\iff$ $\underbrace{u_x \cdot u_y = 0 \ \& \ |u_x| = |u_y|}_{\text{conformal parametrization}}$

Let $u_i \in W^{1,2}(D^2, \mathbb{R}^3)$ be a minimizing sequence for E:

Poincare inequality $\Rightarrow$ $\int_D u_i^2 + |\nabla u_i|^2 \leq C$

Banach-Alaoglu (or Rellich-Kondrakov) $\Rightarrow$ $u_{i_k} \rightharpoonup u \in W^{1,2}$ weakly convergent subsequence

Problem: lack of continuity

i) In general $\lim_{k\to\infty} E(u_{i_k}) \neq E(u)$

But fortunately E is weakly lower semicontinuous, i.e.

$$E(u) \leq \liminf_{k\to\infty} E(u_{i_k}).$$

ii) Boundary condition is in general not preserved under weak convergence, since E is still invariant under the Möbius group

$$G = \left\{ \varphi(z) = e^{i\alpha} \frac{a+z}{1+\bar{a}z} \;\middle|\; \alpha \in \mathbb{R}, |a|<1 \right\}$$

The limit u could be constant!

(resolved by Courant-Lebesgue lemma; see later)

$\rightsquigarrow$ existence of solutions.

Other questions:

- regularity of solutions?

  $u \in W^{1,2}$, $\Delta u = 0$ weakly $\overset{?}{\rightsquigarrow}$ $u \in C^\infty$

  (interior & boundary estimates)

- uniqueness/nonuniqueness?

- stable/unstable solutions?

  mountain pass lemma,

  min-max theory.

First variation / Euler-Lagrange equations

$\Omega \subset \mathbb{R}^n$ bounded, open set with smooth boundary $\partial\Omega$.

$L : \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ smooth function (Lagrangian)

Notation: $L = L(p, z, x)$

$$D_p L = (L_{p_1}, \dots, L_{p_n})$$
$$D_z L = L_z$$
$$D_x L = (L_{x_1}, \dots, L_{x_n})$$

Consider energy functionals of the form

$$E[u] = \int_\Omega L(Du(x), u(x), x)\, dx$$

for smooth $u : \overline{\Omega} \to \mathbb{R}$

with $u = g$ on $\partial\Omega$ (g given function)

Prop If u is a critical point of E, then it satisfies the Euler-Lagrange eqn

$$-\sum_{i=1}^{n} \left(L_{p_i}(Du, u, x) \right)_{x_i} + L_z(Du, u, x) = 0$$

in Ω.

Ex $L(p, z, x) = \frac{1}{2}|p|^2$

$$E[u] = \int_\Omega \frac{1}{2}|Du|^2$$

$$L_{p_i} = p_i \Rightarrow L_{p_i}(Du, u, x) = D_{x_i} u$$

$\Rightarrow$ EL eqn: $\Delta u = 0$.

Proof For any $v \in C_c^\infty(\Omega)$ consider

$$e(\tau) := E[u + \tau v]$$

Then $\dot{e}(0) = 0$.

Compute this explicitly:

$$e(\tau) := \int_\Omega L(Du+\tau Dv,\, u+\tau v,\, x)\, dx$$

$$\Rightarrow \dot{e}(\tau) = \int_\Omega \sum_{i=1}^n L_{p_i}(Du+\tau Dv, u+\tau v, x)\, v_{x_i}\, dx + \int_\Omega L_z(Du+\tau Dv, u+\tau v, x)\, v\, dx$$

$$\Rightarrow 0 = \int_\Omega \sum_{i=1}^n L_{p_i}(Du, u, x)\, v_{x_i} + L_z(Du,u,x)\, v\, dx$$

$$\Rightarrow 0 = \int_\Omega \left(-\sum_{i=1}^n \left(L_{p_i}(Du, u, x)\right)_{x_i} + L_z(Du,u,x)\right) v\, dx$$

$\forall v \in C_c^\infty(\Omega)$ $\Longrightarrow$ Claim $\boxed{\checkmark}$

Ex (generalized Dirichlet principle)

$$L(p,z,x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\, p_i p_j - z f(x)$$

$$a_{ij} = a_{ji}$$

$$E[u] = \int \left(\frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i} u_{x_j} - u f \right) dx$$

$$EL: \quad -\sum_{i,j=1}^{n} \left(a_{ij} u_{x_i}\right)_{x_j} = f$$

Ex (minimal graphs)

$$L(p,z,x) = \left(1 + |p|^2\right)^{1/2}$$

$$E[u] = \int_\Omega \left(1 + |Du|^2\right)^{1/2} dx \quad \text{area of the graph of } u.$$

$$EL: \quad \sum_{i=1}^{n} \left(\frac{u_{x_i}}{\left(1 + |Du|^2\right)^{1/2}} \right)_{x_i} = 0$$

graphical minimal surface eqn (zero mean curvature)

Second variation

$$\ddot{e}(\tau) = \int_\Omega \Big(\sum_{i,j=1}^n L_{p_i p_j}(Du+\tau Dv, u+\tau v, x) v_{x_i} v_{x_j}$$

$$+ 2\sum_{i=1}^n L_{p_i z}(Du+\tau Dv, u+\tau v, x) v_{x_i} v$$

$$+ L_{zz}(Du+\tau Dv, u+\tau v, x) v^2 \Big) dx$$

If u is a minimizer, then $\ddot{e}(0) \geq 0$.

$\Rightarrow$ <u>Stability inequality:</u>

$$Q_u(v) := \int_\Omega \Big(\sum_{i,j=1}^n L_{p_i p_j}(Du, u, x) v_{x_i} v_{x_j}$$

$$+ 2\sum_{i=1}^n L_{p_i z}(Du, u, x) v_{x_i} v$$

$$+ L_{zz}(Du, u, x) v^2 \Big) dx \geq 0.$$

<u>Prop</u> $Q_u(v) \geq 0 \Rightarrow L_{p_i p_j}(Du, u, x) \geq 0$

(positive semidefinite)

Rmks i) This motivates later convexity/ellipticity assumptions.

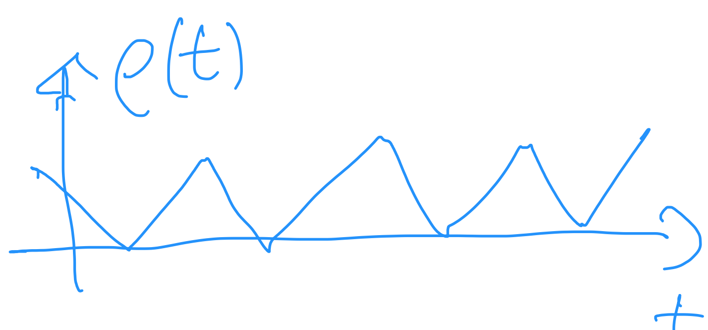
ii) Same is true for Q_u with finite index.

Proof $C_c^\infty(\Omega) \subset W_0^{1,2}(\Omega)$ dense

$$\Rightarrow \quad Q_u(v) \geq 0 \quad \forall v \in W_0^{1,2}(\Omega).$$

Fix $\xi \in \mathbb{R}^n$. Let $v(x) = \varepsilon \rho\left(\frac{x \cdot \xi}{\varepsilon}\right) \chi(x)$

where



periodic zig-zag function

and $\chi \in C_c^\infty(\Omega)$ cutoff function.

$$\Rightarrow \quad v_{x_i}(x) = \rho'\left(\frac{x \cdot \xi}{\varepsilon}\right) \xi_i \chi(x) + O(\varepsilon)$$

$$\Rightarrow \quad 0 \leq \int_\Omega \sum_{ij} L_{p_i p_j}(Du, u, x) \underbrace{\rho'^2}_{=1 \text{ a.e.}} \xi_i \xi_j \chi^2 + O(\varepsilon)$$

$$\varepsilon \to 0 \implies \int_\Omega \sum_{i,j} L_{p_i p_j}(Du, u, x)\,\xi_i \xi_j\, \chi^2 \geq 0$$

$$\forall \chi \in C_c^\infty(\Omega) \implies \sum_{i,j} L_{p_i p_j}(Du, u, x)\,\xi_i \xi_j \geq 0$$

$\blacksquare$