Deformation thm

Assume $E \in C^1(H, \mathbb{R})$ satisfies PS.

Suppose $K_c = \phi$.

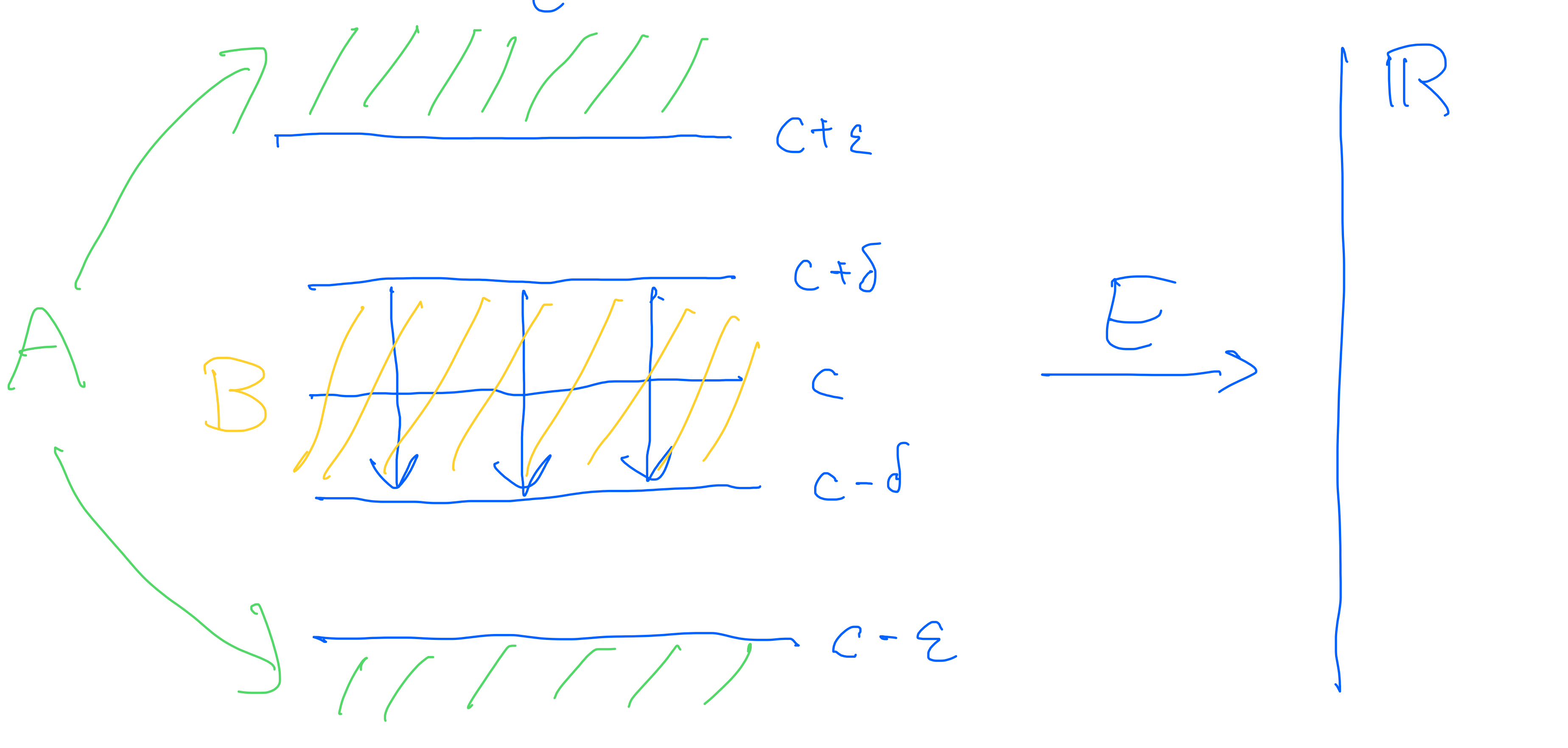
Then $\forall \varepsilon > 0$ $\exists \delta \in (0, \varepsilon)$, $\exists \underset{t \in [0,1]}{\eta_t} : H \to H$ cont,

s.t. (i) $\eta_0 = id$

(ii) $\eta_t(u) = u \quad \forall u \in E^{-1}[c-\varepsilon, c+\varepsilon]$

(iii) $E[\eta_t(u)] \leq E[u]$

(iv) $\eta_1(A_{c+\delta}) \subseteq A_{c-\delta}$



Lemma $\exists\ \sigma, \varepsilon \in (0,1)$ st:

$$\|E'[u]\| \geq \sigma \quad \forall u \in A_{c+\varepsilon} - A_{c-\varepsilon}$$

Proof of Lemma: If not, $\exists\ \sigma_k, \varepsilon_k \to 0$, $\exists\ u_k \in A_{c+\varepsilon_k} \setminus A_{c-\varepsilon_k}$ st:

$$\|E'[u_k]\| \leq \sigma_k$$

$$PS \implies u_{k'} \to u \text{ in } H$$

$$\implies E[u] = c,\ E'[u] = 0$$

$$\implies K_c \neq \emptyset \quad \lightning \qquad \boxed{//}$$

Proof of Deformation Thm (assuming E' loc. Lipschitz)

Fix σ, ε from Lemma.

Fix $\delta \in \left(0, \min\left(\varepsilon, \sigma^2/2\right)\right)$.

Let $A := \{ u \in H \mid E[u] \geq c+\varepsilon \text{ or } E[u] \leq c-\varepsilon \}$

$B := \{ u \in H \mid c-\delta \leq E[u] \leq c+\delta \}$

Idea: $\frac{d}{dt}\eta_t(u) = -\text{suitable factor} \cdot E'[\eta_t(u)]$

$\eta_0(u) = u.$

Let

$$g(u) := \frac{\operatorname{dist}(u, A)}{\operatorname{dist}(u, A) + \operatorname{dist}(u, B)}.$$

$0 \leq g \leq 1, \quad g \equiv 0 \text{ on } A, \; g \equiv 1 \text{ on } B$

Let $h(s) := \begin{cases} 1 & 0 \leq s \leq 1 \\ 1/s & s \geq 1. \end{cases}$

$\Rightarrow V(u) := -g(u) \cdot h(\|E'(u)\|) E'[u]$

is bdd & loc. Lipschitz.

$$\text{ODE} \begin{cases} \dfrac{d\eta_t(u)}{dt} = V(\eta_t(u)) \\ \eta_0(u) = u \end{cases}$$

has unique solution $\eta \in C([0,1] \times H, H)$.

Write $\eta_t(u) = \eta(t, u)$

By construction (i) & (ii) satisfied.

·) $\dfrac{d}{dt} E[\eta_t(u)] \leq 0 \Rightarrow$ (iii).

·) Fix $u \in A_{c+\delta}$. Want to show: $\eta_1(u) \in A_{c-\delta}$.

If $\eta_t(u) \notin B$ for some $t \in [0,1] \Rightarrow$ done.

Suppose towards a contradiction that $\eta_t(u) \in B \;\; \forall t \in [0,1]$

$\Rightarrow g(\eta_t(u)) = 1 \;\; \forall t$.

$$\Rightarrow \frac{d}{dt} E[\eta_t(u)] = \langle E'[\eta_t(u)], V(\eta_t(u)) \rangle$$

$$= -h(\|E'[\eta_t(u)]\|) \, \|E'[\eta_t(u)]\|^2$$

$$\leq -\sigma^2$$

for $\|E'\| \leq 1$ by lemma,
for $\|E'\| \geq 1$ by def of h.

$$\Rightarrow E[\eta_1(u)] \leq E[u] - \sigma^2$$

$$\leq c + \delta - \sigma^2 \leq c - \delta \; \text{\char"21AF}$$

$\boxed{//}$

Recall from last time:

Deform. thm. $\Rightarrow$ Mountain Pass thm.

Mountain Pass Thm

Assume $E \in C^1(H, \mathbb{R})$ satisfies PS.

Suppose (i) $E[0] = 0$

(ii) $\exists r, a > 0$: $E[u] \geq a$ if $\|u\| = r$

(iii) $\exists v \in H$: $\|v\| > r$, $E[v] \leq 0$.

Let $\Gamma := \{ \gamma \in C([0,1], H) \mid \gamma(0) = 0, \gamma(1) = v \}$

Then $c := \inf\limits_{\gamma \in \Gamma} \max\limits_{t \in [0,1]} E[\gamma(t)]$

is a critical value of E.

Application to semilinear elliptic PDEs

$$(*)\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

E.g. $f(u) = |u|^{p-1}u$, $1 < p < \frac{n+2}{n-2}$.

Note: $u \equiv 0$ is a solution.

More generally:

$$|f(z)| \leq C(1+|z|^p)$$

$$|f'(z)| \leq C(1+|z|^{p-1})$$

$$C^{-1}|z|^{p+1} \leq |F(z)| \leq C|z|^{p+1} \qquad F' = f$$

Assume $0 \leq F(z) \leq \gamma z f(z)$ for some $\gamma < 1/2$.

Note: $f(0) = 0 \Rightarrow F(0) = 0$

$\Rightarrow u \equiv 0$ is a solution of $(*)$

Thm (existence)

(*) has a nontrivial (weak) solution.

Proof
$$E[u] := \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)$$

$$u \in H := H_0^1(\Omega) \qquad \text{eg } F(u) = \frac{|u|^{p+1}}{p+1}$$

$$\langle u, v \rangle = \int_\Omega \langle \nabla u, \nabla v \rangle.$$

(i) $E[0] = 0$ since $F(0) = 0$.

(ii) $\|u\| = r$ to be chosen.

$$E[u] = \frac{1}{2} r^2 - \int_\Omega F(u).$$

$$\left|\int_\Omega F(u)\right| \leq C \int_\Omega |u|^{p+1}$$

$$\underset{\text{Hölder}}{\leq} C\left(\int_\Omega |u|^{2^*}\right)^{\frac{p+1}{2^*}} \qquad 2^* = \frac{2n}{n-2}$$

$$\underset{\text{Sobolev}}{\leq} C\|u\|^{p+1} = C r^{p+1}.$$

$$\Rightarrow E[u] \geq \frac{r^2}{4} \text{ for } r \text{ small}.$$

(iii) $u_0 \neq 0$, $v = t u_0$

$$E[v] \leq t^2 \frac{1}{2}\|u_0\|^2 - t^{p+1} C^{-1} \int_\Omega |u_0|^{p+1}$$

$$< 0 \text{ for } t \text{ large}.$$

$\Rightarrow$ By the MPT we can find a nontrivial critical point of E,

provided we check $E \in C^1(H, \mathbb{R})$ & P.S.

$$\circ)\ \langle E'[u], v\rangle = \int_\Omega \nabla E'[u]\cdot\nabla v$$

$$= \frac{d}{dt}\Big|_0 E[u+tv]$$

$$= \frac{d}{dt}\Big|_0 \int_\Omega \left(\frac{1}{2}|\nabla(u+tv)|^2 - F(u+tv)\right)$$

$$= \int_\Omega \nabla u\cdot\nabla v - f(u)\cdot v \qquad (f = F')$$

$$= \int_\Omega \nabla u\cdot\nabla v + \nabla\Delta^{-1}f(u)\,\nabla v$$

$$\Rightarrow \underbrace{E'[u] = u + \Delta^{-1}f(u)}$$

recall: $\left\{\begin{array}{ll} \Delta v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{array}\right\}$ where $g\in H^{-1}(\Omega)$

has a unique solution $v\in H^1_0(\Omega)$

Defines $\Delta^{-1}: H^{-1}(\Omega)\to H^1_0(\Omega)$

Also recall that $f(u) \in L^{\frac{2n}{n+2}}(\Omega) \subseteq H^{-1}(\Omega)$.

$\Rightarrow E \in C^1(H, \mathbb{R})$ (cf Evans for details)

Check PS: Suppose $u_k \in H = H_0^1$

$$|E[u_k]| \leq C$$

$$E'[u_k] = u_k + \Delta^{-1} f(u_k) \to 0 \text{ in } H_0^1$$

want to find convergent subsequence.

Let $\varepsilon > 0$. Then for k large enough:

$$|\langle E'[u_k], v \rangle| = \left| \int_\Omega \nabla u_k \nabla v - f(u_k) v \right|$$

$$= \left| \int_\Omega \nabla (u_k + \Delta^{-1} f(u_k)) \cdot \nabla v \right|$$

$$\leq \varepsilon \|v\| \qquad (v \in H_0^1(\Omega))$$

Choose $v = u_k$

$$\Rightarrow \left| \int_\Omega |\nabla u_k|^2 - f(u_k)u_k \right| \le \varepsilon \|u_k\|$$

$$\underset{\varepsilon=1}{\Rightarrow} \int_\Omega f(u_k)u_k \le \|u_k\|^2 + \|u_k\|$$

On the other hand:

$$E[u_k] = \frac{1}{2}\|u_k\|^2 - \int_\Omega F(u_k) \le C.$$

$$\Rightarrow \|u_k\|^2 \le C + 2\int_\Omega F(u_k)$$

$$\le C + 2\underbrace{\gamma}_{<1}\left(\|u_k\|^2 + \|u_k\|\right)$$

$2\gamma < 1 \Rightarrow u_k$ is bdd in $H_0^1(\Omega)$.

$\Rightarrow \quad u_{k'} \rightharpoonup u$ weakly in $H_0^1(\Omega)$

$\quad\quad\;\; \to u$ in $L^{p+1}(\Omega)$.

(since $p+1 < 2^*$).

$\Rightarrow f(u_{k'}) \to f(u)$ in $H^{-1}(\Omega)$

$\Rightarrow \Delta^{-1} f(u_{k'}) \to \Delta^{-1} f(u)$ in $H_0^1(\Omega)$.

Since $u_k + \Delta^{-1} f(u_k) \longrightarrow 0$ in $H_0^1(\Omega)$ by assumption,

we conclude that

$u_{k'} \longrightarrow u$ in $H_0^1(\Omega)$

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