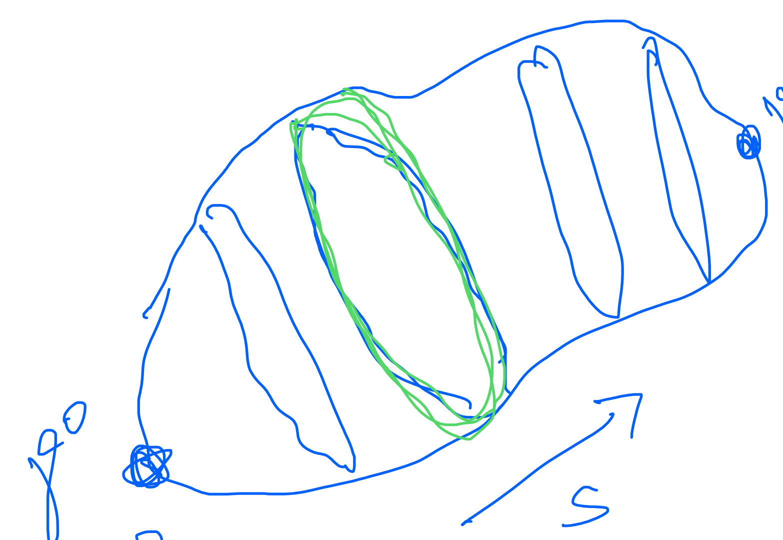
Min-Max theory

Q: Can we find critical points, that are not minima?

Motivation (S^2, g) ← arbitrary Riem. metric

Thm (Birkhoff)
$\exists$ nontrivial closed geodesic.



Proof idea $E[\gamma] = \frac{1}{2}\int_0^{2\pi} |\dot{\gamma}|^2 dt, \; \gamma : S^1 \to S^2$

Sweepout $\{\gamma_t^s\}_{s,t\in[0,1]}$

$$\inf_{\{\gamma_t^s\}\ \text{sweepout}} \max_{s\in[0,1]} E[\gamma^s] \rightsquigarrow \text{critical point of index } 1.$$

General setup

$(H, \langle \cdot, \cdot \rangle)$ Hilbert space, $\|u\| = \sqrt{\langle u, u \rangle}$.

$E : H \to \mathbb{R}$

Def: E is differentiable at $u \in H$ if $\exists v \in H$:

$$E[w] = E[u] + \langle v, w-u \rangle + o(\|w-u\|)$$

Notation: $v = E'[u]$.

Def: $\cdot)$ $E \in C^1(H, \mathbb{R})$ if $E'[u]$ exist $\forall u \in H$ and $E' : H \to H$, $u \mapsto E'[u]$ is continuous

$\cdot)$ $E \in C^{1,1}_{loc}(H, \mathbb{R})$ if in addition $u \mapsto E'[u]$ is locally Lipschitz.

Rmk: Theory works for C^1, but some proofs simplify for $C^{1,1}$.

Def: $E \in C^1(H,\mathbb{R})$ satisfies the <u>Palais-Smale compactness condition</u> (P.S.) if each sequence $u_k \in H$ s.t.

(i) $E[u_k]$ is bounded,

(ii) $E'[u_k] \to 0$ in H,

has a convergent subsequence.

<u>Ex</u> Every $E \in C^1(\mathbb{R}^n, \mathbb{R})$ with $\lim_{\|u\|\to\infty} |E[u]| + |E'[u]| = \infty$ satisfies (PS).

<u>Ex</u> $E : H^1(S^1, M) \to \mathbb{R}$

$E[\gamma] = \frac{1}{2}\int_0^{2\pi} |\dot{\gamma}|^2 dt$ satisfies (P.S.)

Critical points / deformations

For $c \in \mathbb{R}$ let

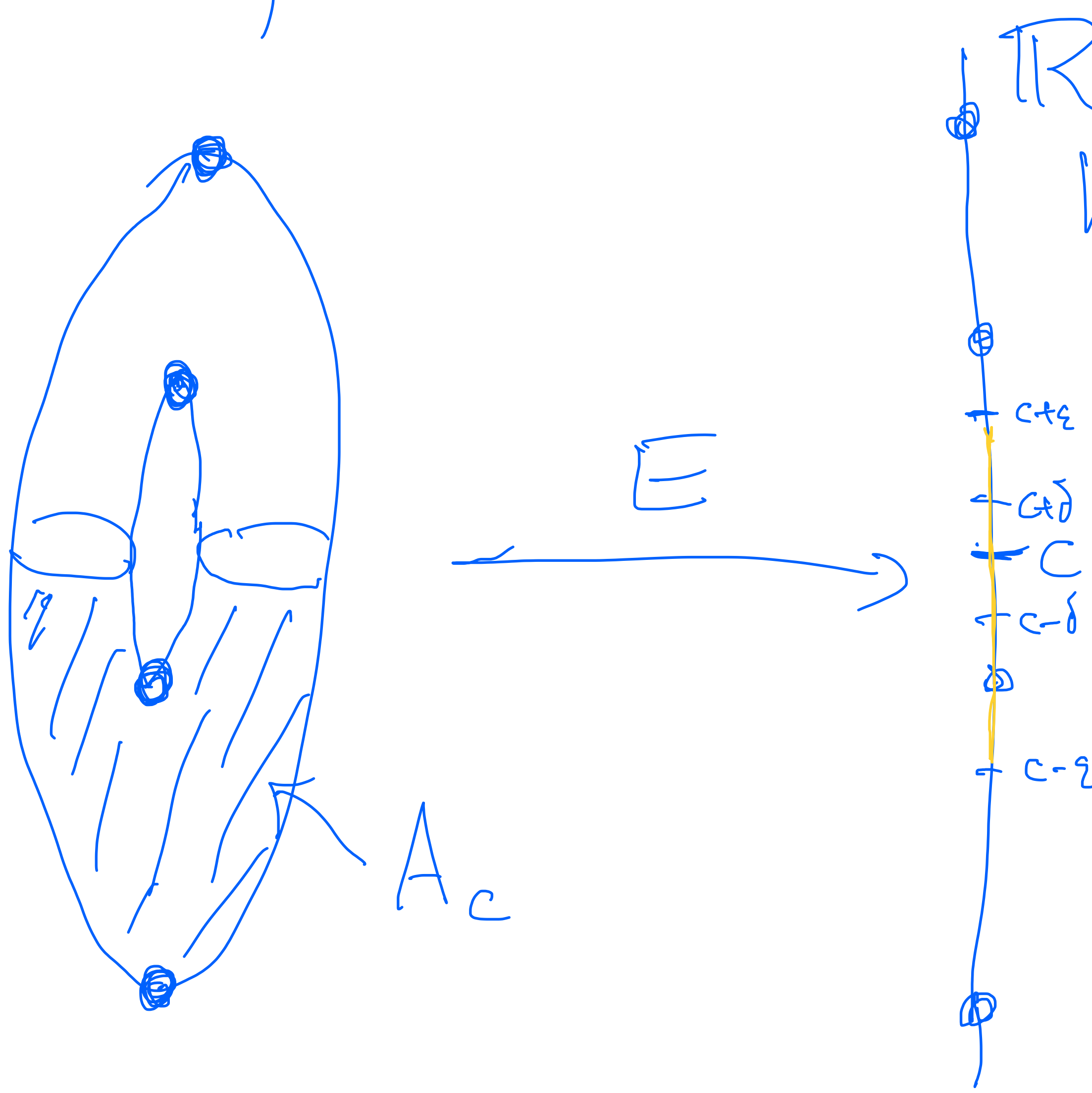
$$A_c = \{ u \in H \mid E[u] \le c \}$$

$$K_c = \{ u \in H \mid E[u] = c,\ E'[u] = 0 \}$$

Def .) $u \in H$ is a critical point if $E'[u] = 0$.

.) $c \in \mathbb{R}$ is a critical value if $K_c \neq \phi$.

"Ex"



height function of torus has 4 critical points of index 0, 1, 1, 2.

Idea: If c is not a critical value, we can deform $A_{c+\varepsilon}$ into $A_{c-\varepsilon}$.

Deformation thm

Assume $E \in C^1(H, \mathbb{R})$ satisfies PS and suppose $K_c = \phi$.

Then for each sufficiently small ε $\exists \delta \in (0, \varepsilon)$, $\exists \eta \in C([0,1] \times H, H)$ s.t. $\eta_t(u) = \eta(t, u)$ satisfies:

(i) $\eta_0(u) = u \quad \forall u \in H$

(ii) $\eta_1(u) = u \quad \forall u \notin E^{-1}[c-\varepsilon, c+\varepsilon]$

(iii) $E[\eta_t(u)] \leq E[u] \quad \forall u \in H, \forall t \in [0,1]$

(iv) $\eta_1(A_{c+\delta}) \subseteq A_{c-\delta}$.

Moutain Pass thm

Assume $E \in C^1(H, \mathbb{R})$ satisfies PS.

Suppose also

(i) $E[0] = 0$

(ii) $\exists\, r, a > 0$ s.t. $E[u] \geq a$ if $\|u\| = r$

(iii) $\exists\, v \in H$ with $\|v\| > r$, $E[v] \leq 0$.

Define $\Gamma = \left\{ \gamma \in C([0,1], H) \;\middle|\; \begin{array}{l} \gamma(0) = 0 \\ \gamma(1) = v \end{array} \right\}$

Then $c := \inf\limits_{\gamma \in \Gamma} \max\limits_{t \in [0,1]} E[\gamma(t)]$

is a critical value of E.

Proof 1) Clearly $c \geq a$.

2) Suppose towards a contradiction c is not a critical value, i.e. $K_c = \emptyset$.

Def. Thm. $\Longrightarrow$ for $\varepsilon \in (0, \frac{a}{2})$ small, $\exists\, \delta \in (0, \varepsilon)$, $\exists\, \eta : H \to H$ cont. with

$$\eta(A_{c+\delta}) \subseteq A_{c-\delta}$$

and $\eta(u) = u \quad \forall u \notin E^{-1}[c-\varepsilon, c+\varepsilon]$.

3) Select $\gamma \in \Gamma$ with $\max\limits_{t\in[0,1]} E[\gamma(t)] \leq c+\delta$.

Then $\hat{\gamma} := \eta \circ \gamma \in \Gamma$,

and $\max\limits_{t\in[0,1]} E[\hat{\gamma}(t)] \leq c-\delta$

$\Longrightarrow c \leq c - \delta$ ↯ $\boxed{\text{q.e.d.}}$