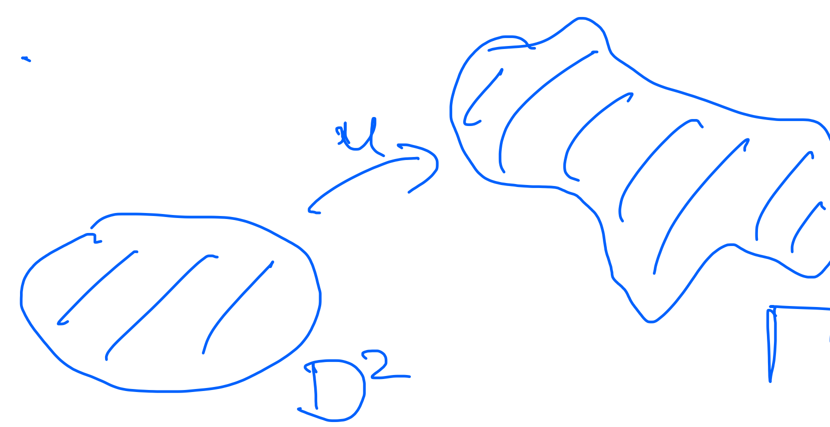
Plateau problem

$\Gamma \subset \mathbb{R}^n$ simple closed curve (say C^1).

Want to find disc of minimal area with boundary Γ.

For $u \in H^1(D^2, \mathbb{R}^n)$ we consider



$$A[u] = \int_D \sqrt{|u_x|^2 |u_y|^2 - (u_x \cdot u_y)^2}\, dx\, dy$$

$$E[u] = \frac{1}{2} \int_D \left(|u_x|^2 + |u_y|^2\right) dx\, dy$$

Note that $A[u] \leq E[u]$

with "=" $\iff \begin{cases} |u_x| = |u_y| \\ u_x \cdot u_y = 0 \end{cases}$ a.e.

i.e. u is weakly conformal.

Morrey's ε-conformality lemma

Given $u \in H^1(D^2, \mathbb{R}^n)$, $\varepsilon > 0$, $\exists \varphi : D^2 \to D^2$ diffeomorphism s.t. $\tilde{u} := u \circ \varphi$ satisfies

$$E[\tilde{u}] \leq (1+\varepsilon) A[\tilde{u}].$$

Proof see Morrey's Annals '48 paper.

Now, consider

$$\mathcal{F}_\Gamma := \left\{ u \in H^1(D^2, \mathbb{R}^n) \;\middle|\; \begin{array}{l} u|_{\partial D^2} \in C^0(\partial D^2, \mathbb{R}^n) \\ \text{is a monotone} \\ \text{parametrization} \\ \text{of } \Gamma \end{array} \right\}$$

Def $\gamma : S^1 \to \Gamma$ is called a monotone parametrization of Γ if γ is onto & $\gamma^{-1}(V)$ is connected $\forall V \subset \Gamma$ connected.

Morrey's lemma $\Rightarrow$ $\displaystyle \inf_{u \in \mathcal{F}_\Gamma} A[u] = \inf_{u \in \mathcal{F}_\Gamma} E[u] =: m_\Gamma.$

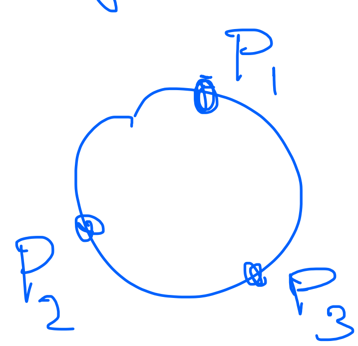
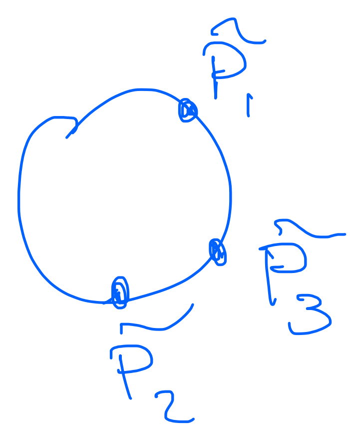
Issue (cf HW): $\mathcal{F}_\Gamma$ is not closed under weak convergence:

If $\varphi \in \mathcal{G} := \left\{ \varphi(z) = e^{i\alpha} \frac{a+z}{1+\bar{a}z} \;\middle|\; \begin{matrix} \alpha \in \mathbb{R}/2\pi\mathbb{Z} \\ |a|<1 \end{matrix} \right\}$,

then $E[u \circ \varphi] = E[u]$.

Now consider sequence φ_i where $a_i \to 1$.

Lemma from complex analysis

Given  & 

$\exists! \varphi \in \mathcal{G}$ s.t. $\varphi(P_j) = \tilde{P}_j$ for $j = 1, 2, 3$.

Now, fix $Q_1, Q_2, Q_3 \in \Gamma$, let $P_j = e^{2\pi i j/3}$,

set $\mathcal{F}_\Gamma^* := \{ u \in \mathcal{F}_\Gamma \mid u(P_j) = Q_j \text{ for } j = 1,2,3 \}$

still have $m_\Gamma = \inf_{u \in \mathcal{F}_\Gamma^*} A[u] = \inf_{u \in \mathcal{F}_\Gamma^*} E[u]$.

Key Lemma $\mathcal{F}^*_\Gamma \to C^0(\partial D^2, \mathbb{R}^n)$ is compact.

Proof: Later.

Thm (Douglas | Rado)

Let $\Gamma \subset \mathbb{R}^n$ be a simple closed curve (say C^1). Then there exists $u : D^2 \to \mathbb{R}^n$ st.

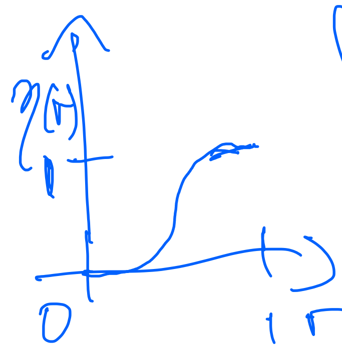
(1) $u|_{\partial D^2} : \partial D^2 \to \Gamma$ is a monotone parametrization of Γ.

(2) $u \in H^1(D) \cap C^0(\bar{D}) \cap C^\infty(D)$

(3) $u(D^2)$ minimizes area among all maps from the disc with boundary value Γ.

<u>Proof</u> $\cdot$) $\mathcal{F}_\Gamma^* \neq \emptyset$: $\gamma: S^1 \to \Gamma$ monotone parametrization with $\gamma(P_j) = Q_j$.

Choose



& set $u_0(r,\theta) := \eta(r)\gamma(\theta)$.

Then $u_0 \in \mathcal{F}_\Gamma^*$.

$\cdot$) Let:

$u_i \in \mathcal{F}_\Gamma^*$ be a minimizing sequence,

i.e. $E[u_i] \longrightarrow m_\Gamma$.

$$\Longrightarrow \quad u_i - u_0 \in H_0^1(D^2, \mathbb{R}^4)$$

Poincaré $\Longrightarrow$ $\underbrace{\int_D (u_i - u_0)^2}_{= \int u_i^2 + \int u_0^2 - \underbrace{2\int u_i u_0}_{\le \frac{1}{2}\int u_i^2 + 2\int u_0^2}} \le \underbrace{\int_D |\nabla(u_i - u_0)|^2}_{\le 2\int |\nabla u_i|^2 + 2\int |\nabla u_0|^2}$

$$\Longrightarrow \|u_i\|_{H^1} \le C.$$

$\Rightarrow u_i \rightharpoonup u$ weakly in $H^1(D^2, \mathbb{R}^n)$ for a subsequence.

Key lemma $\Rightarrow$ $u_i|_{\partial D} \to u|_{\partial D}$ uniformly, in particular $u|_{\partial D}$ is continuous & a monotone parametrization of Γ $\Rightarrow$ (1).

·) By construction $A[u] = E[u] = m_\Gamma \Rightarrow$ (3)

·) Can assume $u_i \in C^0(\bar{D}) \cap C^2(D)$ and $\Delta u_i = 0$ in D.

Otherwise replace u_i by v_i sol. of

$$\begin{cases} \Delta v_i = 0 & \text{in } D \\ v_i = u_i & \text{on } \partial D \end{cases}$$

$\Rightarrow v_i \in C^2(D) \cap C^0(\bar{D}) \cap H^1(D)$,

$$E[v_i] \le E[u_i].$$

Then limit u satisfies

$$\begin{cases} \Delta u = 0 \\ |u_x| = |u_y|,\ u_x \cdot u_y = 0 \end{cases} \quad \text{in } D.$$

Max. principle $\Rightarrow$

$$\sup_D |u_i - u_j| \le \max_{\partial D} |u_i - u_j|$$

$\Rightarrow u_i \to u \in C^0(\bar{D})$ uniformly.

Std estimates $\Rightarrow u \in C^\infty(D) \Rightarrow (2)_{[1]}$

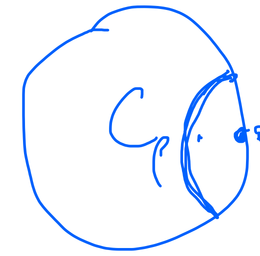
Key remaining step:

<u>Courant-Lebesgue lemma</u>

For any $u \in H^1(D^2)$ any $p \in \bar{D}$,

any $\delta \in (0,1)$, $\exists\, \rho \in [\delta, \delta^{1/2}]$ st if s denotes arclength on

$C_\rho = C_\rho(p) = \partial B_\rho(p) \cap D$



we have $u_s \in L^2(C_\rho)$ and

$$\int_{C_\rho} |u_s^2| \, ds \le \frac{8E[u]}{\rho|\ln\rho|}.$$

Proof By Fubini $u_s \in L^2(C_\rho)$ for a.e. $\rho < 1$, and

$$2E[u] \ge \int_{(B_{\sqrt{\delta}}(p) \setminus B_\delta(p)) \cap D} |\nabla u|^2 \ge \int_\delta^{\sqrt{\delta}} \int_{C_\rho} |u_s|^2 \, ds \, d\rho$$

$$\ge \operatorname*{ess\,inf}_{\delta \le \rho \le \sqrt{\delta}} \left(\rho \int_{C_\rho} |u_s|^2 ds \right) \cdot \underbrace{\int_\delta^{\sqrt{\delta}} \frac{d\rho}{\rho}}_{\ge \frac{1}{2|\ln\rho|} \text{ for all } \rho \in [\delta, \sqrt{\delta}].}.$$

$\Rightarrow$ Claim

Finally: $u_i \in \mathcal{J}_P^*$, $u_i \to u$ weakly in H^1

$\Rightarrow$ u_i bdd in H^1.

C.L. lemma $\Rightarrow$ $u_i|_{\partial D} \to u|_{\partial D}$ uniformly.

(fill in details yourself:
Use C.L. lemma to show
that $u_i|\partial D$ is uniformly continuous).

Remarks on regularity

·) regularity of map u:
If $\Gamma \in C^{k,\alpha}$ then $u \in C^{k,\alpha}(\bar{D})$
(Hildebrandt)

·) regularity of the image $u(\bar{D})$:

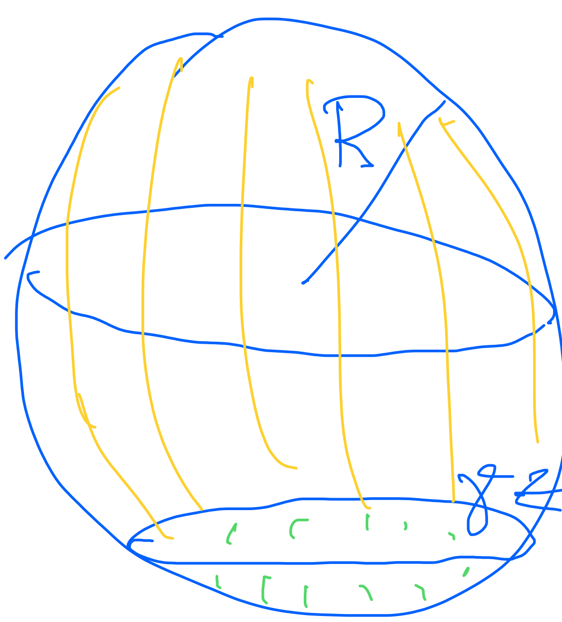
Def $x \in \bar{D}$ is a branch point if $\nabla u|_x = 0$

Γ analytic, $n=3 \Rightarrow$ no branch points (Osserman).

Variants:

$$\Delta u = 2H u_x \wedge u_y$$

$H = \text{const}$ (mean curvature)



$H = \frac{1}{R}$

γ bounds two $H = \frac{1}{R}$ surfaces a small & large one.

Rellich conjecture $\gamma \subset B_R$.

For every $H \in (0, 1/R)$ there exist at least two H-surfaces with boundary γ.

proved in 80s via mountain pass arguments (see Struwe's book).

·) other topologies,
in manifolds / metric spaces,
in higher dimensions ($\rightarrow$ geometric measure theory)