$$(*)\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

last lecture: existence of nontrivial solutions for $1 < p < \frac{n+2}{n-2}$.

(<u>subcritical case</u>)

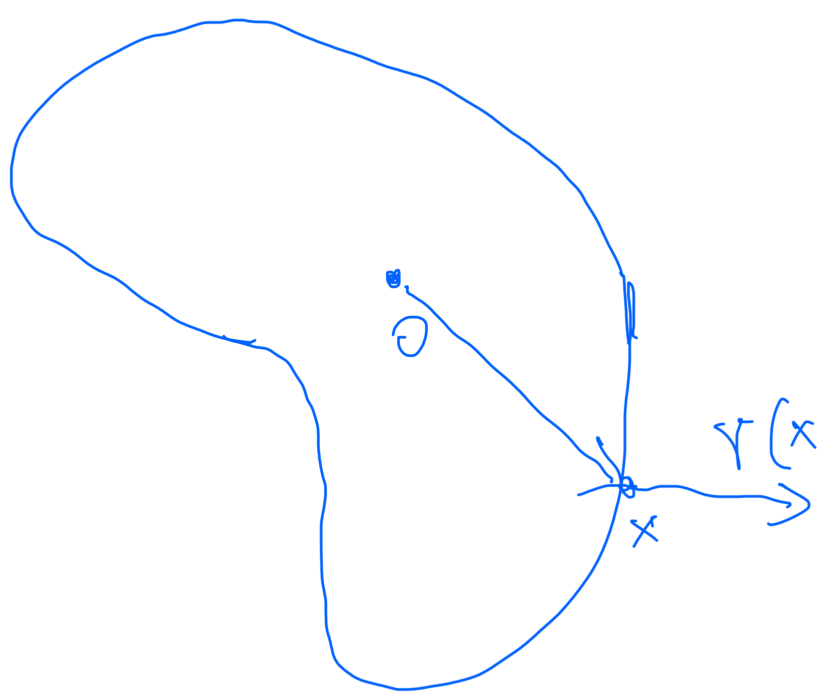
recall: $H_0^1(\Omega) \subset L^{2^*}(\Omega)$,

$$2^* = \frac{2n}{n-2} = \frac{n+2}{n-2} + 1,$$

the <u>critical Sobolev exponent</u>

today: nonexistence result for <u>supercritical case</u> $\left(p > \frac{n+2}{n-2}\right)$

Def A C^1-domain $\Omega \subset \mathbb{R}^n$ is **starshaped** wrt 0 provided $\forall x \in \overline{\Omega}$ the line segment $\{\lambda x \mid 0 \le \lambda \le 1\} \subseteq \overline{\Omega}$



Rmk .) Ω starshaped wrt $0 \Rightarrow x \cdot \nu(x) \geq 0$ $\forall x \in \partial\Omega$.

.) Ω convex, $0 \in \Omega \Rightarrow \Omega$ starshaped wrt 0
$\not\Leftarrow$

Thm (nonexistence)

Assume Ω is starshaped wrt 0.

Assume $p > \frac{n+2}{n-2}$.

If $u \in C^2(\bar{\Omega})$ solves $(*)$, then $u \equiv 0$.

Proof $\quad -\Delta u = |u|^{p-1} u$ $\quad$ multiply by $x \cdot \nabla u$ and integrate

$$\Rightarrow \int_\Omega (-\Delta u)(x \cdot \nabla u) = \int_\Omega |u|^{p-1} u \, (x \cdot \nabla u)$$

$$\text{LHS} = -\sum_{ij} \int_\Omega u_{x_i x_i} \, x_j \, u_{x_j}$$

$$= \underbrace{\sum_{ij} \int_\Omega u_{x_i} (x_j u_{x_j})_{x_i}} - \underbrace{\sum_{ij} \int_{\partial\Omega} u_{x_i} \nu^i x_j u_{x_j}}_{= \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu)}$$

$\nabla u = \pm |\nabla u| \nu$ on $\partial\Omega$

$$L_3 = \sum_{ij} \int_\Omega \left(u_{x_i} \delta_{ij} u_{x_j} + u_{x_i} x_j u_{x_j x_i} \right)$$

$$= \int_\Omega \left(|\nabla u|^2 + \sum_j \left(\frac{|\nabla u|^2}{2} \right)_{x_j} x_j \right)$$

$$= \left(1 - \frac{n}{2}\right) \int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \frac{|\nabla u|^2}{2} (x \cdot \nu)$$

$$\Rightarrow LHS = \frac{2-n}{2} \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu)$$

$$RHS = \sum_j \int_\Omega |u|^{p-1} u x_j u_{x_j}$$

$$= \sum_j \int_\Omega \left(\frac{|u|^{p+1}}{p+1} \right)_{x_j} x_j$$

$$= -\frac{n}{p+1} \int_\Omega |u|^{p+1}$$

$$\Rightarrow \boxed{\frac{n-2}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 (x\cdot\nu) = \frac{n}{p+1}\int_\Omega |u|^{p+1}}$$

(Pohozaev identity)

Since Ω is starshaped wrt 0, this yields

$$\frac{n-2}{2}\int_\Omega |\nabla u|^2 \leq \frac{n}{p+1}\int_\Omega |u|^{p+1}$$

On the other hand,

$$-\Delta u = |u|^{p-1}u \quad \Big| \cdot u, \text{ IBP}$$

$$\Rightarrow \int_\Omega |\nabla u|^2 = \int_\Omega |u|^{p+1}$$

$$\Rightarrow \left(\frac{n-2}{2} - \frac{n}{p+1}\right)\int_\Omega |u|^{p+1} \leq 0$$

Since $p > \frac{n+2}{n-2} \Rightarrow u \equiv 0$ $\boxed{}$

Critical case $p = \frac{n+2}{n-2}$.

We could try to minimize

$$E[u] = \int_\Omega |\nabla u|^2$$

Subject to the constraint $\int_\Omega |u|^{2^*} = 1$

Or equivalently we could try to minimize the Sobolev quotient

$$S[u] = \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{2^*}\right)^{2/2^*}} \qquad S[\lambda u] = S[u]$$

Def: $S(\Omega) := \inf_{0 \neq u \in H^1_0(\Omega)} S[u]$

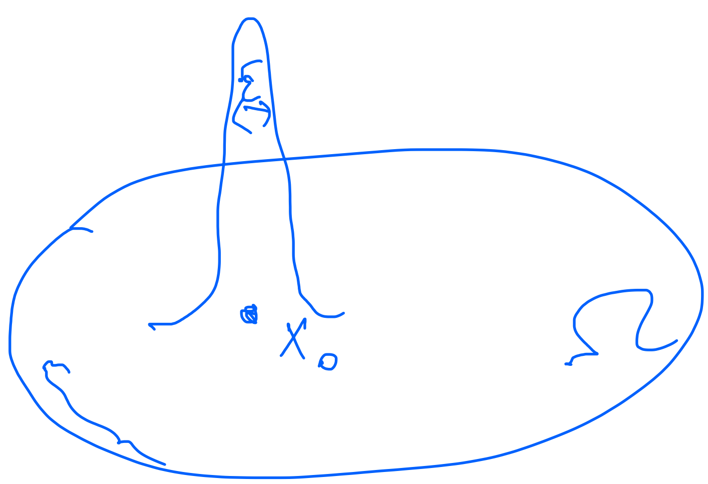
Rmk $S(\Omega)^{-1/2}$ is the optimal constant for the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$.

(in particular $S(\Omega) > 0$).

$$\left[\left(\int_\Omega |\nabla u|^2 \right)^{1/2} S(\Omega)^{-1/2} \geq \left(\int |u|^{2^*} \right)^{1/2^*} \right]$$

$\Omega = \mathbb{R}^n$, Sobolev extremal function $u(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}}$.

also: $u_{(x_0,\varepsilon)}(x) = \left(\frac{\varepsilon}{\varepsilon^2+|x-x_0|^2}\right)^{\frac{n-2}{2}}$.



$\rightsquigarrow$ •) $S(\Omega) = S(\mathbb{R}^n)$ independent of Ω

•) S is never attained on any $\Omega \subsetneq \mathbb{R}^n$.