Lagrange multipliers

① Nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Idea: Minimize

$$E[u] = \frac{1}{2}\int_\Omega |Du|^2 \qquad (u=0 \text{ on } \partial\Omega)$$

subject to the constraint

$$F[u] = \int_\Omega G(u) = 0 \qquad (g = G')$$

$$\mathcal{A} := \{ u \in H_0^1(\Omega) \mid F[u] = 0 \}$$

technical assumption: $|g(x)| \leq C(1+|x|)$.

Prop (existence) If $\mathcal{A} \neq \emptyset$, then $\exists\, u \in \mathcal{A}$ s.t. $E[u] = \min_{w \in \mathcal{A}} E[w]$.

Proof Let $u_k \in \mathcal{A}$ be a minimizing sequence:

$$E[u_k] \to m := \inf_{w \in \mathcal{A}} E[w].$$

$\Rightarrow \exists$ subsequence $u_{k'}$:

$$u_{k'} \rightharpoonup u \text{ weakly in } H_0^1(\Omega)$$

$$u_{k'} \to u \text{ strongly in } L^2(\Omega).$$

l.s.c. $\Rightarrow E[u] \leq m$ (hence "=").

$$|F[u]| = |F[u] - \underbrace{F[u_k]}_{=0}| \leq \int_\Omega |G(u) - G(u_k)|$$

$$\leq C \int_\Omega |u - u_k| (1 + |u| + |u_k|) \xrightarrow{k \to \infty} 0 \quad \boxed{\text{Ez.}}$$

Thm (Lagrange multiplier)

Suppose $u \in \mathcal{A}$ satisfies $E[u] = \min_{w \in \mathcal{A}} E[w]$.

Then $\exists \lambda \in \mathbb{R}$ s.t.

$$\int_\Omega Du \cdot Dv = \lambda \int_\Omega g(u)\, v \quad \forall v \in H_0^1(\Omega).$$

Proof If $\underbrace{g(u) = 0 \text{ a.e. in } \Omega}$,

then $u \equiv 0$, λ arbitrary is a solution.

Suppose now, not.

$$\Rightarrow \exists w \in H_0^1(\Omega) : \int_\Omega g(u)\, w \neq 0.$$

Now consider

$$f(\tau, \sigma) := F[u + \tau v + \sigma w]$$

$$= \int_\Omega G(u + \tau v + \sigma w)$$

Note that:

$$f(0,0) = 0$$

$$\frac{\partial f}{\partial \tau} = \int_\Omega g(u+\tau v+\sigma w)\,v$$

$$\frac{\partial f}{\partial \sigma} = \int_\Omega g(u+\tau v+\sigma w)\,w$$

Since $\frac{\partial f}{\partial \sigma}(0,0) = \int_\Omega g(u)\,w \neq 0$,

by implicit fn thm $\exists\, \phi$ C^1 with $\phi(0)=0$

s.t. $f(\tau, \phi(\tau)) = 0$. (for $|\tau| \ll 1$)

$$\Rightarrow \frac{\partial f}{\partial \tau}(\tau,\phi(\tau)) + \frac{\partial f}{\partial \sigma}(\tau,\phi(\tau))\cdot\dot{\phi}(\tau) = 0$$

$$\Rightarrow \dot{\phi}(0) = -\frac{\int_\Omega g(u)\,v}{\int_\Omega g(u)\,w}.$$

Now set $\quad w(\tau) := \tau v + \phi(\tau) w$

$$e(\tau) := E[\underbrace{u + w(\tau)}_{\in \mathcal{A}}]$$

$$\Rightarrow 0 = \dot{e}(0) = \int_\Omega Du \cdot D\dot{w}(0)$$

$$= \int_\Omega Du \cdot (Dv + \dot{\phi}(0) Dw)$$

$$\Rightarrow \int_\Omega Du \cdot Dv = -\dot{\phi}(0) \int_\Omega Du \cdot Dw$$

$$= \underbrace{\frac{\int_\Omega Du Dw}{\int_\Omega g(u) w}}_{=: \lambda} \cdot \int_\Omega g(u) v$$

$\boxed{///}$

(II) Harmonic maps

pointwise constraint: $|u| = 1$

Minimize $E[u] = \frac{1}{2}\int_\Omega |Du|^2$

over the admissible class

$$\mathcal{A} := \left\{ u \in H^1(\Omega; \mathbb{R}^m) \;\middle|\; \begin{array}{l} u = g \text{ on } \partial\Omega \\ |u| = 1 \text{ a.e.} \end{array} \right\}$$

i.e. we are minimizing energy among maps $u : \Omega \to S^{m-1} \subset \mathbb{R}^m$

(with $u = g$ on $\partial\Omega$)

given boundary condition.

Prop (existence) If $\mathcal{A} \neq \emptyset$, then

$\exists u \in \mathcal{A}$ s.t. $E[u] = \min\limits_{w \in \mathcal{A}} E[w]$

Proof $u_k \in \mathcal{A}$ min. sequence,

i.e. $E[u_k] \to m = \inf\limits_{w \in \mathcal{A}} E[w]$

$$\Rightarrow \begin{cases} u_{k'} \longrightarrow u \text{ weakly in } H^1 \\ u_{k'} \longrightarrow u \text{ strongly in } L^2 \\ u_{k'}(x) \to u(x) \text{ for a.e. } x \end{cases}$$

$\Rightarrow |u| = 1$ a.e.

trace thm $\Rightarrow$ $u = g$ on $\partial\Omega$

$\Rightarrow u \in \mathcal{A}$

And $E[u] = m$ $\boxed{//}$

Thm (EL eqn for harmonic maps)

If $u \in \mathcal{A}$ satisfies $E[u] = \min_{w \in \mathcal{A}} E[w]$

then $\int_\Omega \langle Du, Dv \rangle = \int_\Omega |Du|^2 \langle u, v \rangle$

for all $v \in H^1_0(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$

Rmk This means that $u = (u^1, \dots, u^m)$ is a weak solution of

$$\begin{cases} -\Delta u = |Du|^2 u & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

In particular, the Lagrange multiplier $\lambda = |Du|^2$ is a function (since we had the pointwise constraint $|u| = 1$)

<u>Proof</u> Fix $v \in H^1_0 \cap L^\infty$.

Note that $|u + \tau v| \neq 0$ a.e. for $|\tau| \ll 1$

$\Rightarrow$ $v(\tau) := \dfrac{u + \tau v}{|u + \tau v|} \in \mathcal{A}$

$\Rightarrow$ Setting $e(\tau) := E[v(\tau)]$

we have $\dot{e}(0) = 0$.

Compute

$$0 = \dot{e}(0) = \int_\Omega \langle Du, D\dot{v}(0) \rangle$$

$$\dot{v}(0) = v - \langle u, v \rangle u$$

$$\Rightarrow \int_\Omega \langle Du, Dv \rangle = \int_\Omega \underbrace{\langle Du, D(\langle u, v \rangle u) \rangle}$$

$$= |Du|^2 \langle u, v \rangle + \underbrace{\langle Du, (D\langle u, v \rangle) u \rangle}_{= 0}$$

since for unit vector its rate of change is always perpendicular to itself.

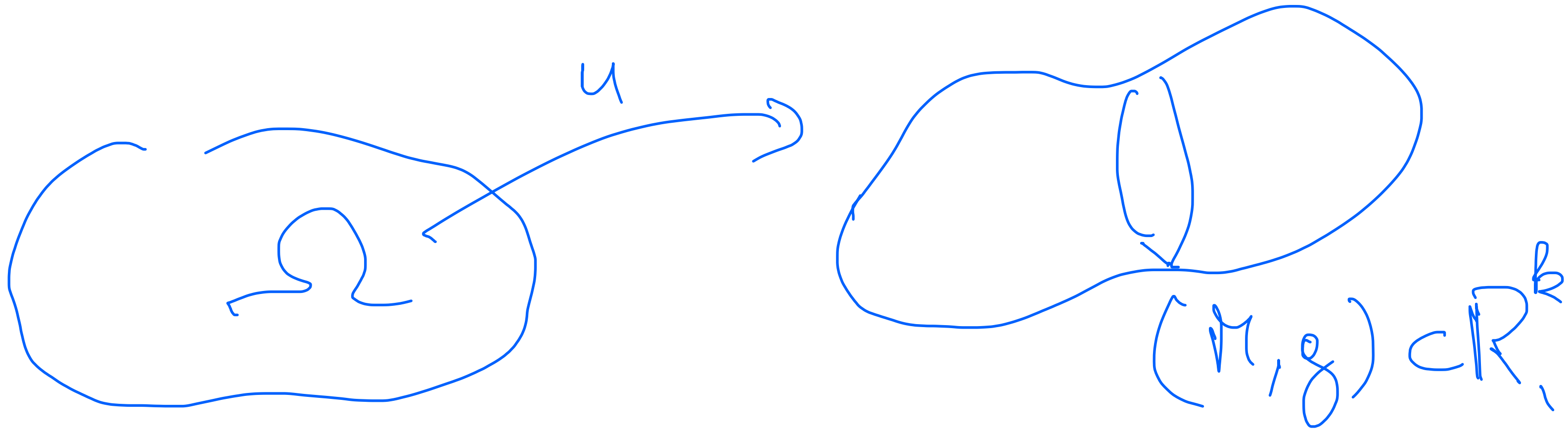
$$\Rightarrow \int_\Omega \langle Du, Dv\rangle = \int_\Omega |Du|^2 \langle u, v\rangle \qquad \boxed{/\!/}$$

More generally:

target (M, g) closed Riem. mfd.

Nash $\Rightarrow$ $(M^m, g) \xhookrightarrow{\text{isom}} \mathbb{R}^k$ for some large k.

$$\mathcal{A} := \left\{ u \in H^1(\Omega; \mathbb{R}^k) \;\middle|\; \begin{array}{l} u = g \text{ on } \partial\Omega \\ u(x) \in M \text{ a.e. } x \in \Omega \end{array} \right\}$$



$$E[u] = \frac{1}{2}\int_\Omega |\nabla u|^2, \quad |\nabla u|^2 = \sum_{i=1}^{n} \sum_{a=1}^{k} (\partial_i u^a)^2$$

Euler-Lagrange eqn?

Admissible $v \in C^\infty_c(\Omega; \mathbb{R}^k)$ with $v(x) \in T_{u(x)}M$ for all $x \in \Omega$.

$$\Rightarrow \quad 0 = \frac{d}{d\tau}\Big|_{\tau=0} E[u+\tau v]$$

$$= -\int_\Omega \underbrace{\langle \Delta u, v \rangle}_{= \sum_{i=1}^n \sum_{a=1}^k \partial_i \partial_i u^a v^a}$$

$$\Rightarrow \quad -\Delta u(x) \perp T_{u(x)}M \quad \forall x \in \Omega$$

$$\Rightarrow \quad -\Delta u(x) = \sum_{\alpha=m+1}^k \lambda_\alpha(x)\, \nu_\alpha(u(x))$$

where $\nu_{m+1}, \dots, \nu_k$ is a local ON frame for the normal bundle $TM^\perp$ near $u(x)$.

Compute Lagrange multipliers $\lambda_\alpha(x)$:

$$\lambda_\alpha = -\langle \Delta u, \nu_\alpha \circ u \rangle$$

$$= -\underbrace{\operatorname{div}\langle \nabla u, \nu_\alpha \circ u \rangle}_{=0 \text{ since } \partial_i u \in T_u M} + \langle \nabla u, (d\nu_\alpha \circ u) \cdot \nabla u \rangle$$

$$\Rightarrow \boxed{-\Delta u = A(u)(\nabla u, \nabla u)}$$

(harmonic map equ)

where $A(u) : T_u M \times T_u M \to T_u^\perp M$

is the 2^{nd} fundamental form,

$$A(p)(X, Y) = \sum_{\alpha=m+1}^{k} A_\alpha(p)(X, Y)\, \nu_\alpha(p)$$

$$A_\alpha(p)(X, Y) = \langle X, d\nu_\alpha Y \rangle.$$

Q: Given $u_0 : \Omega \longrightarrow (M, g)$

can we deform it to a harmonic map?

Harmonic map heat flow (Eells-Sampson)

$$\begin{cases} \partial_t u = \Delta u + A(u)(\nabla u, \nabla u) \\ u\big|_{t=0} = u_0 \end{cases}$$