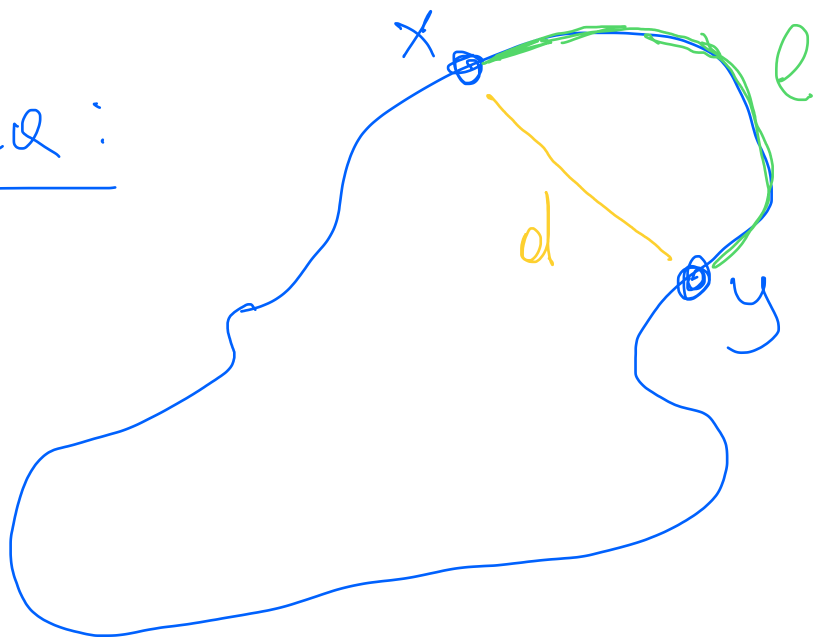
Huisken's distance comparison principle

<u>Idea</u>:



d extrinsic distance

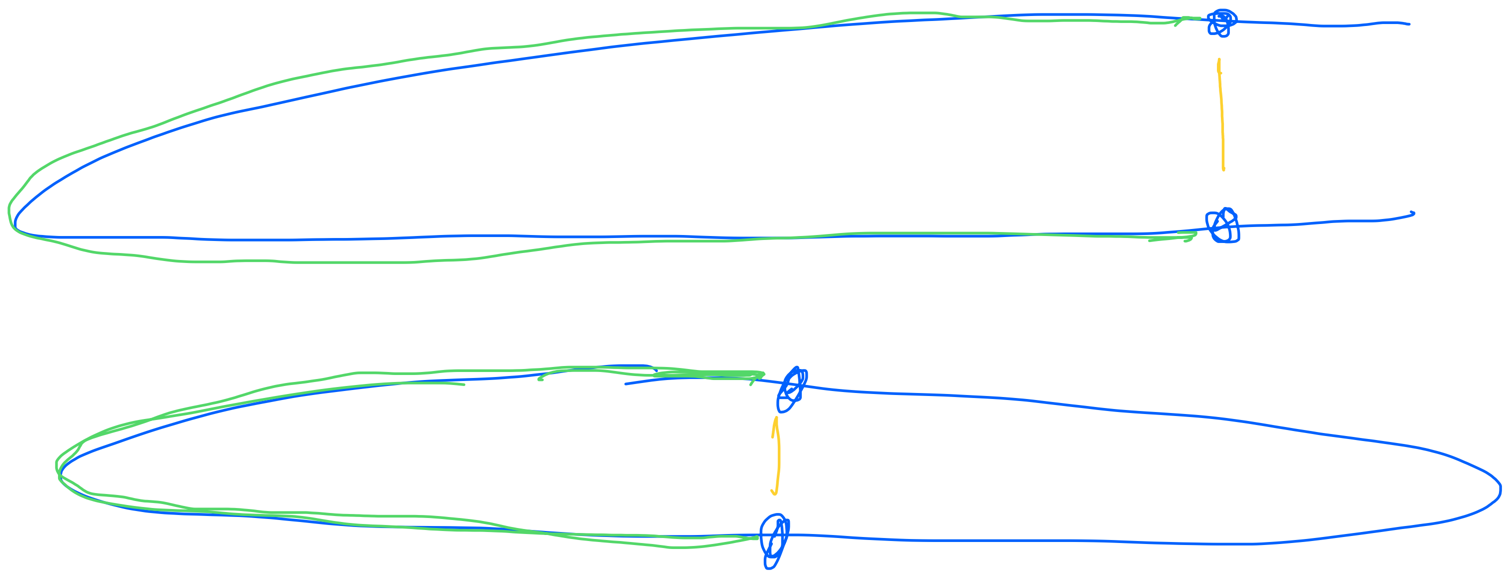
ℓ intrinsic distance

roughly speaking $\frac{\ell}{d} \le C$

is preserved under CSF.

"quantitative version of the fact that embeddedness is preserved under CSF".

<u>Main application</u>: Grim reaper & paperclip cannot arise as blowup limit of CSF of closed embedded curves



$X : S^1 \times [0,T) \to \mathbb{R}^2$ CSF

$L(t) :=$ total length at time t.

$x, y \in S^1$:

$\ell(x,y,t) =$ intrinsic distance between $X(x,t)$ & $X(y,t)$ at time t.

$d(x,y,t) = |X(x,t) - X(y,t)|$.

Consider :

$$\boxed{R(t) := \sup_{x \neq y} \frac{L(t)}{\pi d(x,y,t)} \sin \frac{\pi \ell(x,y,t)}{L(t)}}$$

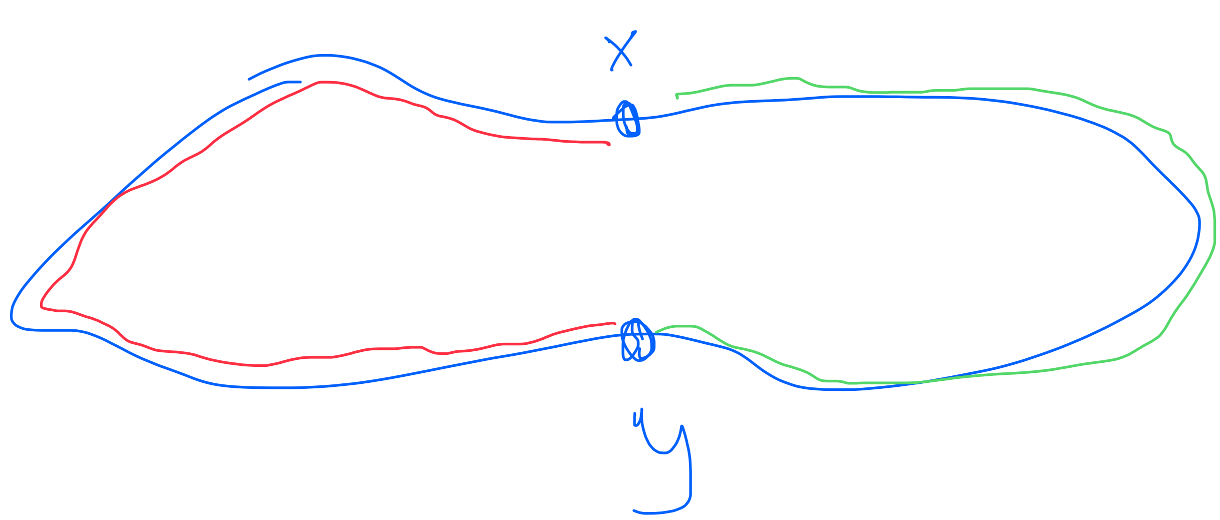
Rmks .) for $\ell \ll L$ we get

$$\frac{L}{\pi d}\sin\frac{\pi \ell}{L} \approx \frac{\ell}{d}.$$

$\uparrow$ $\sin x \approx x$ for $|x| \ll 1$

.) $R \geq 1$ and $R = 1 \iff$ round circle.

.) Since $\sin\left(\frac{\pi}{2}+\varphi\right) = \sin\left(\frac{\pi}{2}-\varphi\right)$ the function $\sin\frac{\pi \ell(x,y,t)}{L(t)}$ is smooth even at points with $\ell(x,y,t) = L(t)/2$.



Thm (Huisken's distance comparison principle)

If a family of closed embedded curves in the plane evolves by CSF, then

$$R(t) = \sup_{x \neq y} \frac{L(t)}{\pi d(x,y,t)} \sin \frac{\pi \ell(x,y,t)}{L(t)}$$

is nonincreasing in time.

Proof If not, $\exists t_0 < t_1$, $\exists r > 1$:

$$R(t_0) < r, \quad R(t_1) > r.$$

Consider the function

$$Z(x,y,t) := r \cdot d(x,y,t) - \frac{L(t)}{\pi} \sin \frac{\pi \ell(x,y,t)}{L(t)}.$$

$\Rightarrow \exists \bar{t} \in (t_0, t_1)$, $\exists \bar{x} \neq \bar{y}$

s.t.: $Z(\bar{x}, \bar{y}, \bar{t}) = 0$, $Z(x,y,t) \geq 0$ $\forall x, y \in S^1$, $\forall t \in (t_0, \bar{t})$.

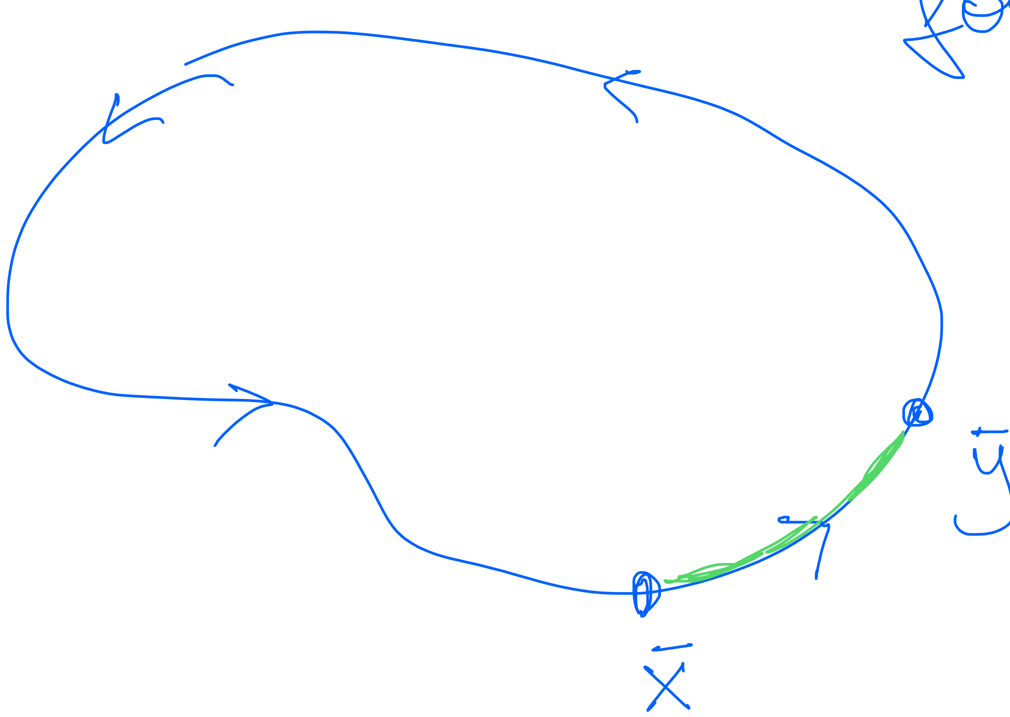
goal: Want to get contradiction with maximum principle.

have to compute $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x \partial y}$, $\dfrac{\partial^2 z}{\partial y^2}$, $\dfrac{\partial z}{\partial t}$.

Setup: wlog assume parametrization at time $\bar{t}$ is by arclength, orientation is st

$$\partial_x \ell(x,y,\bar{t}) = -1, \quad \partial_y \ell(x,y,\bar{t}) = +1$$

for x close to $\bar{x}$, y close to $\bar{y}$.



$$Z(x,y,t) := r\,d(x,y,t) - \frac{L(t)}{\pi} \sin \frac{\pi \ell(x,y,t)}{L(t)}$$

Compute:

$\cdot)$ $$\frac{\partial Z}{\partial x}(x,y,\bar{t}) = r \frac{\left\langle X(x,\bar{t}) - X(y,t), \frac{\partial X}{\partial x}(x,\bar{t}) \right\rangle}{|X(x,\bar{t}) - X(y,\bar{t})|} + \cos \frac{\pi \ell(x,y,\bar{t})}{L(\bar{t})}.$$

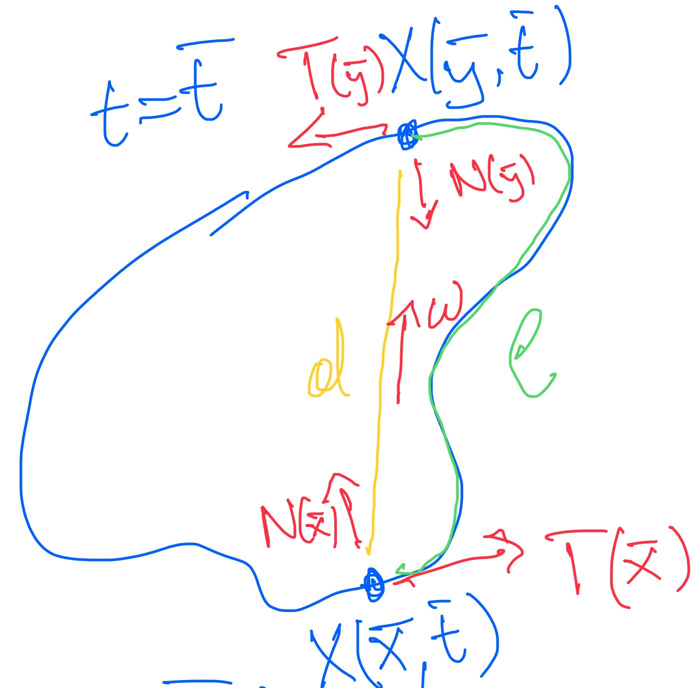
$\cdot)$ $$\frac{\partial Z}{\partial y}(x,y,\bar{t}) = -r \frac{\left\langle X(x,\bar{t}) - X(y,\bar{t}), \frac{\partial X}{\partial y}(y,\bar{t}) \right\rangle}{|X(x,\bar{t}) - X(y,\bar{t})|} - \cos \frac{\pi \ell(x,y,\bar{t})}{L(\bar{t})}.$$

These 1^{st} derivatives vanish when evaluated at $(\bar{x},\bar{y},\bar{t})$.

In particular, adding gives:

$$0 = \left\langle X(\bar{x},\bar{t}) - X(\bar{y},\bar{t}),\ \frac{\partial X}{\partial x}(\bar{x},\bar{t}) - \frac{\partial X}{\partial y}(\bar{y},\bar{t}) \right\rangle \quad (*)$$

Notation:



$$T(\bar{x}) = \frac{\partial X}{\partial x}(\bar{x},\bar{t})$$

$$T(\bar{y}) = \frac{\partial X}{\partial y}(\bar{y},\bar{t})$$

$$K(\bar{x})N(\bar{x}) = \frac{\partial^2 X}{\partial x^2}(\bar{x},\bar{t})$$

$$K(\bar{y})N(\bar{y}) = \frac{\partial^2 X}{\partial^2 y}(\bar{y},\bar{t})$$

$$d = d(\bar{x},\bar{y},\bar{t})$$

$$\ell = \ell(\bar{x},\bar{y},\bar{t})$$

$$L = L(\bar{t})$$

$$w = \frac{X(\bar{y},\bar{t}) - X(\bar{x},\bar{t})}{d}$$

In this notation, $(*)$ becomes

$$\langle w, T(\bar{x})\rangle = \langle w, T(\bar{y})\rangle .$$

Final exam: Mo, April 19^{th}, 10am – 1pm.

Compute x-derivative of

$$\frac{\partial z}{\partial x} = r\,\frac{\langle X(\cancel{x},t)-X(y,t), \frac{\partial X}{\partial x}(\cancel{x},t)\rangle}{|X(\cancel{x},t)-X(y,t)|} + \cos\frac{\pi\,\ell(\cancel{x},y,t)}{L(t)}$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2}(\bar{x},\bar{y},\bar{t}) = \frac{r}{d}\underbrace{\langle T(\bar{x}), T(\bar{x})\rangle}_{=1} + r\langle -w, \kappa(\bar{x})N(\bar{x})\rangle$$

$$-\frac{r}{d}\langle w, T(\bar{x})\rangle^2 + \frac{\pi}{L}\sin\frac{\pi\ell}{L}$$

$$= \frac{r}{d}\left(1-\langle w, T(\bar{x})\rangle^2\right)$$

$$-r\kappa(\bar{x})\langle w, N(\bar{x})\rangle + \frac{\pi}{L}\sin\frac{\pi\ell}{L}.$$

Similarly:

$\cdot)$ $$\frac{\partial^2 z}{\partial y^2}(\bar{x},\bar{y},\bar{t}) = \frac{r}{d}\left(1-\langle w, T(\bar{y})\rangle^2\right) + r\kappa(\bar{y})\langle w, N(\bar{y})\rangle + \frac{\pi}{L}\sin\frac{\pi \ell}{L}$$

$\cdot)$ $$\frac{\partial^2 z}{\partial x \partial y}(\bar{x},\bar{y},\bar{t}) = -\frac{r}{d}\left(\langle T(\bar{x}), T(\bar{y})\rangle - \langle T(\bar{x}), w\rangle\langle w, T(\bar{y})\rangle\right) - \frac{\pi}{L}\sin\frac{\pi \ell}{L}.$$

Define $\alpha \in (0, \pi/2)$ by

$$\cos\alpha = \langle w, T(\bar{x})\rangle = \langle w, T(\bar{y})\rangle$$

Then $\langle T(\bar{x}), T(\bar{y})\rangle = \cos(2\alpha)$.

Now sum up to obtain:

$$\frac{\partial^2 z}{\partial x^2}(\bar{x},\bar{y},\bar{t})+\frac{\partial^2 z}{\partial y^2}(\bar{x},\bar{y},\bar{t})-2\frac{\partial^2 z}{\partial x \partial y}(\bar{x},\bar{y},\bar{t})$$

$$=-r\kappa(\bar{x})\langle \omega, N(\bar{x})\rangle + r\kappa(\bar{y})\langle \omega, N(\bar{y})\rangle$$

$$+\frac{4\pi}{L}\sin\frac{\pi \ell}{L}.$$

Finally :

$$\frac{\partial z}{\partial t}(\bar{x},\bar{y},\bar{t}) = -r\langle \omega, \kappa(\bar{x})N(\bar{x})-\kappa(\bar{y})N(\bar{y})\rangle$$

$$+\left(\frac{1}{\pi}\sin\frac{\pi\ell}{L}-\frac{\ell}{L}\cos\frac{\pi\ell}{L}\right)\int_{S^1}\kappa^2$$

$$+\cos\frac{\pi\ell}{L}\int_{\bar{x}}^{\bar{y}}\kappa^2 .$$

Since $r>1$, $z(\bar{x},\bar{y},\bar{t})=0$, the curve $X(S^1,\bar{t})$ cannot be round, thus:

$$\int_{S^1} \kappa^2 > \frac{1}{L}\left(\int_{S^1} \kappa\right)^2 = \frac{4\pi^2}{L}$$

$$\int_{\bar{x}}^{\bar{y}} \kappa^2 \geq \frac{1}{\ell}\left(\int_{\bar{x}}^{\bar{y}} \kappa\right)^2 = \frac{4\alpha^2}{\ell}.$$

Conclusion:

$$0 \geq \frac{\partial z}{\partial t}(\bar{x},\bar{y},\bar{t}) - \frac{\partial^2 z}{\partial x^2}(\bar{x},\bar{y},\bar{t}) - \frac{\partial^2 z}{\partial y^2}(\bar{x},\bar{y},\bar{t}) + 2\frac{\partial^2 z}{\partial x \partial y}(\bar{x},\bar{y},\bar{t})$$

$$> \frac{4}{\ell}\left(\alpha^2 - \frac{\pi^2 \ell^2}{L^2}\right)\cos\frac{\pi \ell}{L}.$$

On the other hand:

$$r > 1 \Rightarrow \cos\alpha \leq \cos\frac{\pi \ell}{L}$$

$$\Rightarrow \alpha \geq \frac{\pi \ell}{L} \quad \lightning$$

Derivative test from calculus:

$$\partial_t Z \le 0$$

positive semi-definite

$$M := \begin{pmatrix} \partial_x^2 Z & \partial_x \partial_y Z \\ \partial_x \partial_y Z & \partial_y^2 Z \end{pmatrix} \ge 0$$

$$v^T M v \ge 0 \quad \forall v \in \mathbb{R}^2$$

Apply this for $v = (-1, 1)$ or so, to get

$$\partial_x^2 Z + \partial_y^2 Z - 2 \partial_x \partial_y Z \ge 0 .$$