Moser Iteration and ε-regularity

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The goal of these notes is to explain step by step how to prove the following ε-regularity theorem via Moser iteration:

Theorem
There are constants $\varepsilon > 0$ and $C < \infty$ with the following significance. If $n = 4$ and $u \in H^1(B_1)$ is a nonnegative function that satisfies

$$-\triangle u \leq u^2 \tag{1}$$

weakly, then

$$\|u\|_{L^2(B_1)} \leq \varepsilon \qquad \Rightarrow \qquad \|u\|_{L^\infty(B_{1/2})} \leq C\|u\|_{L^2(B_1)}. \tag{2}$$

First step of the Moser iteration. Choose a cutoff-function $0 \leq \eta \leq 1$ that is 1 on $B_{3/4}$, has support in B_1, and satisfies $|\nabla \eta| \leq 8$. Multiplying (1) by $\eta^2 u$ and integrating by parts we obtain

$$\int_{B_1} \eta^2 |\nabla u|^2 \leq 2\int_{B_1} \eta |\nabla u||\nabla \eta| u + \int_{B_1} \eta^2 u^3. \tag{3}$$

Dealing with the first term on the right hand side by Young's inequality and absorption, this gives the estimate

$$\tfrac{1}{2}\int_{B_1} \eta^2 |\nabla u|^2 dV \leq 128 \int_{B_1} u^2 + \int_{B_1} \eta^2 u^3. \tag{4}$$

For the last term, using Hölder's inequality, the assumption that the energy on B_1 is less than ε, and the Sobolev-inequality, we get

$$\begin{aligned} \int_{B_1} \eta^2 u^3 &\leq \left(\int_{B_1} u^2\right)^{1/2} \left(\int_{B_1} (\eta u)^4\right)^{1/2} \\ &\leq \varepsilon C_S^2 \int_{B_1} |\nabla(\eta u)|^2 \\ &\leq 2\varepsilon C_S^2 \int_{B_1} \eta^2 |\nabla u|^2 + 128 \varepsilon C_S^2 \int_{B_1} u^2, \end{aligned} \tag{5}$$

where $C_S < \infty$ is the local Sobolev constant on B_1. The main idea is that if we choose ε so small that $2\varepsilon C_S^2 \leq \frac{1}{4}$ then the $\int \eta^2 |\nabla u|^2$ term can be absorbed, giving

$$\tfrac{1}{4}\int_{B_1} \eta^2 |\nabla u|^2 \leq 144 \int_{B_1} u^2 \tag{6}$$

and using the Sobolev inequality we arrive at the L^4-estimate

$$\|u\|_{L^4(B_{3/4})} \leq 24 C_S \|u\|_{L^2(B_1)}. \tag{7}$$

General step of the Moser iteration. Fix $\varepsilon \leq \frac{1}{8} C_S^{-2}$, where C_S is the local Sobolev constant on B_1. Consider the sequence of radii $r_k = \frac{1}{2} + \frac{1}{2^k}$ interpolating between $r_1 = 1$ and $r_\infty = \frac{1}{2}$. We want to prove by induction an estimate of the form

$$\|u\|_{L^{2^{k+1}}(B_{r_{k+1}})} \leq C_k \|u\|_{L^{2^k}(B_{r_k})}. \tag{8}$$

The case $k = 1$ has already been established above (with $C_1 = 24 C_S$). For general $k \geq 2$ we multiply (1) by $\eta_k^2 u^{2\alpha_k - 1}$, where $\alpha_k = 2^{k-1}$, and $0 \leq \eta_k \leq 1$ is a cutoff function that equals 1 on $B_{r_{k+1}}$, has support in B_{r_k}, and satisfies $|\nabla \eta_k| \leq 2/(r_k - r_{k+1})$. After integration by parts, we obtain

$$\frac{2\alpha_k - 1}{\alpha_k^2} \int_{B_{r_k}} \eta_k^2 |\nabla u^{\alpha_k}|^2 \leq 2 \int_{B_{r_k}} \eta_k |\nabla u| |\nabla \eta_k| u^{2\alpha_k - 1} + \int_{B_{r_k}} \eta_k^2 u^{2\alpha_k + 1}. \tag{9}$$

Dealing with the first term on the right hand side by Young's inequality and absorption, this gives the estimate

$$\frac{2\alpha_k - 1}{2\alpha_k^2} \int_{B_{r_k}} \eta_k^2 |\nabla u^{\alpha_k}|^2 \leq \frac{32\alpha_k^2}{2\alpha_k - 1} \int_{B_{r_k}} u^{2\alpha_k} + \int_{B_{r_k}} \eta_k^2 u^{2\alpha_k + 1}. \tag{10}$$

For the last term, using Hölder's inequality, the Peter-Paul inequality, the Sobolev-inequality, and the estimate (7), we compute

$$\begin{aligned} \int_{B_{r_k}} \eta_k^2 u^{2\alpha_k+1} &\leq \left(\int_{B_{r_k}} \eta_k^4 u^{4\alpha_k} \right)^{1/4} \left(\int_{B_{r_k}} \eta_k^4 u^4 \right)^{1/4} \left(\int_{B_{r_k}} u^{2\alpha_k} \right)^{1/2} \\ &\leq \delta_k C_S^2 \int_{B_{r_k}} |\nabla(\eta_k u^{\alpha_k})|^2 + \frac{1}{4\delta_k} \left(\int_{B_{r_k}} \eta_k^4 u^4 \right)^{1/2} \int_{B_{r_k}} u^{2\alpha_k} \\ &\leq 2\delta_k C_S^2 \int_{B_{r_k}} \eta_k^2 |\nabla u^{\alpha_k}|^2 + \left(128 \delta_k C_S^2 \alpha_k^2 + \frac{1}{4\delta_k} 24^2 C_S^2 \varepsilon^2 \right) \int_{B_{r_k}} u^{2\alpha_k}. \end{aligned} \tag{11}$$

Choosing $\delta_k = (2\alpha_k - 1)/(8\alpha_k^2 C_S^2)$ the first term can be absorbed, giving

$$\frac{2\alpha_k - 1}{4\alpha_k^2} \int_{B_{r_k}} \eta_k^2 |\nabla u^{\alpha_k}|^2 \leq \left(\frac{32\alpha_k^2}{2\alpha_k - 1} + 16(2\alpha_k - 1) + \frac{18\alpha_k^2}{2\alpha_k - 1} \right) \int_{B_{r_k}} u^{2\alpha_k}, \tag{12}$$

and using the Sobolev inequality we arrive at the estimate (8), with

$$C_k \leq (D 2^{2k})^{1/2^k}, \tag{13}$$

where $D < \infty$ is a universal constant (in fact $D = 100$ works). The product of the constants C_k is bounded and sending $k \to \infty$ gives the desired estimate

$$\|u\|_{L^\infty(B_{1/2})} \leq C \|u\|_{L^2(B_1)}. \tag{14}$$