PDE II - Final Exam (April 19, 2021)

Robert Haslhofer

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, and let $f : \overline{\Omega} \to \mathbb{R}$ be a smooth function. Consider the energy functional

$$E[u] := \int_\Omega |\nabla u|^2 + fu, \qquad u \in H^1_0(\Omega). \tag{0.1}$$

- Find the Euler-Lagrange equation.
- Prove that there exists a unique minimizer.

2. Consider the equation

$$-\partial_i \left(a_{ij} \partial_j u \right) = f \tag{0.2}$$

on the 4-torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$. Assume that $f \in L^2(\mathbb{T}^4)$ and that $a_{ij} \in L^\infty(\mathbb{T}^4)$ and that there exists a constant $\theta > 0$ such that

$$a_{ij}(x)\xi^i\xi^j \geq \theta|\xi|^2, \qquad \forall x \in \mathbb{T}^4, \xi \in \mathbb{R}^4. \tag{0.3}$$

Prove that there exists a constant $C < \infty$ such that

$$\|u\|_{L^4(\mathbb{T}^4)} \leq C \left(\|u\|_{L^2(\mathbb{T}^4)} + \|f\|_{L^2(\mathbb{T}^4)} \right). \tag{0.4}$$

3. Let $u(x,t)$, where $x \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and $t \geq 0$, be a smooth solution of the heat equation $\partial_t u = \triangle u$. Prove that

$$\max_{x\in\mathbb{T}^n} |\nabla u|(x,t_2) \leq \max_{x\in\mathbb{T}^n} |\nabla u|(x,t_1), \qquad \forall t_2 \geq t_1. \tag{0.5}$$

4. Let $\{\Gamma_t \subset \mathbb{R}^2\}_{t\in[0,T)}$ be a curve shortening flow of closed embedded curves in the plane, defined on a maximal time interval $[0,T)$. Let $L(t)$ be the length, and $A(t)$ be the enclosed area. Consider the isoperimetric difference $I(t) := L^2(t) - 4\pi A(t)$.

- Prove that $\frac{d}{dt} I(t) \leq 0$.
- Prove that $\lim_{t\nearrow T} I(t) = 0$.
- Conclude that Γ_0 satisfies the isoperimetric inequality $L^2 \geq 4\pi A$.