Flow through neck-singularities and open problems

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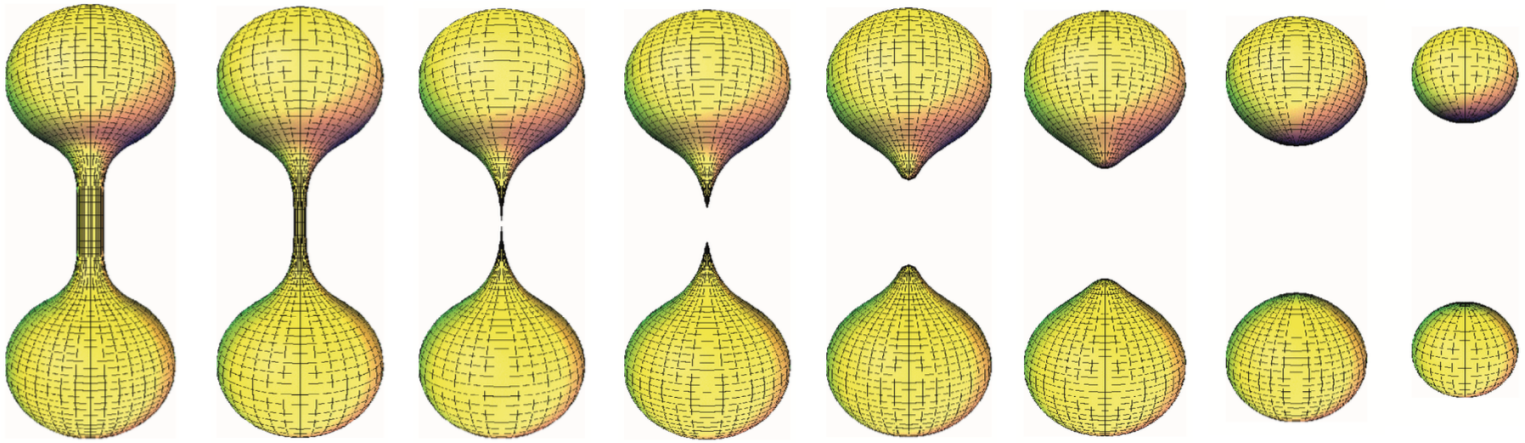
Mean curvature flow

Recall that, given a surface $M_0 \subset \mathbb{R}^3$, we evolve it by

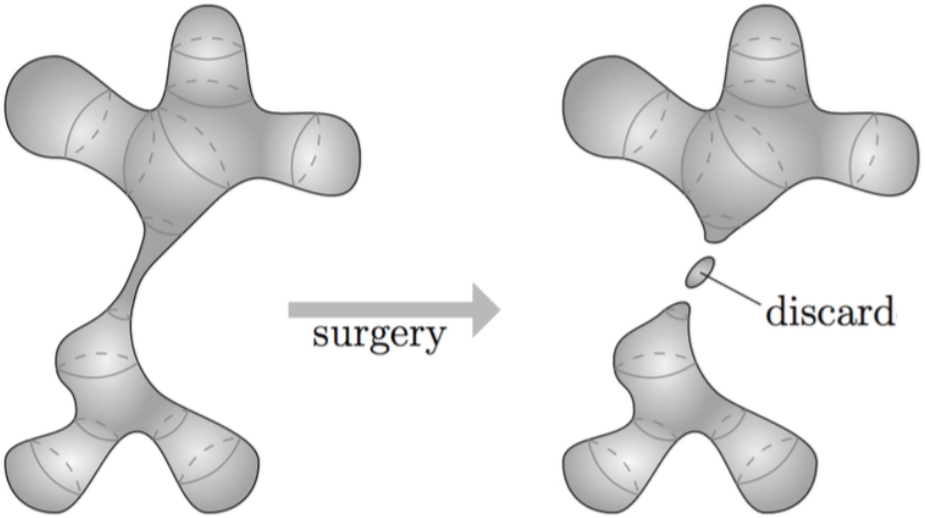
$$\partial_t x = \vec{H}(x) \qquad (x \in M_t)$$



This can be viewed as geometric version of the heat flow $\partial_t u = \Delta u$, but MCF is nonlinear. Hence, the crucial problem is to analyze singularities.

Mean-convex case ($\vec{H} = H\vec{\nu}, H > 0$)

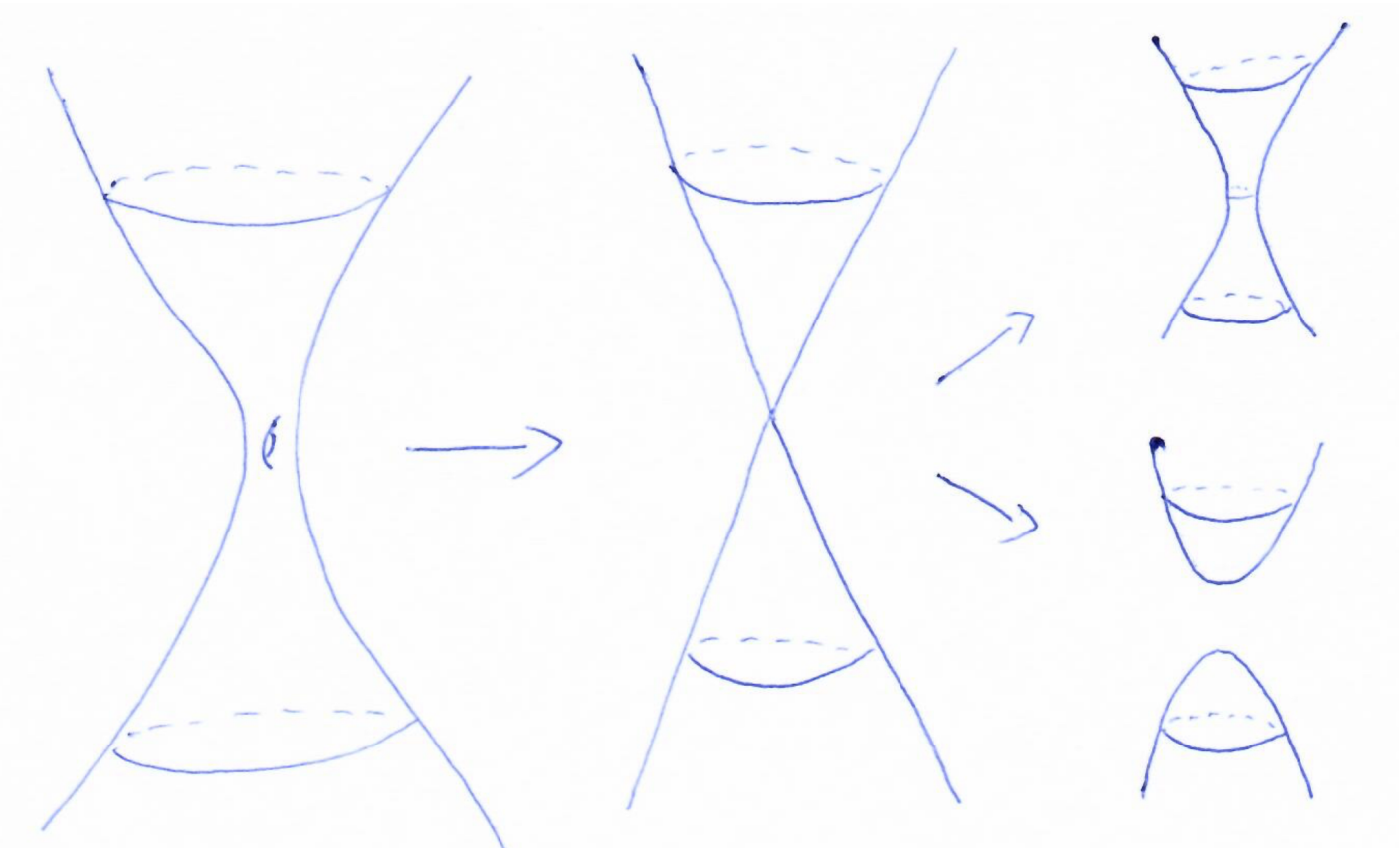
1. unique weak solution
   - flow uniquely through all singularities
   - small singular set (in particular smooth at a.e. time)
   - tangent flows: planes, spheres or cylinders
2. flow with surgery



White, Huisken-Sinestrari, Brendle-Huisken, H-Kleiner

General case - Much less understood!

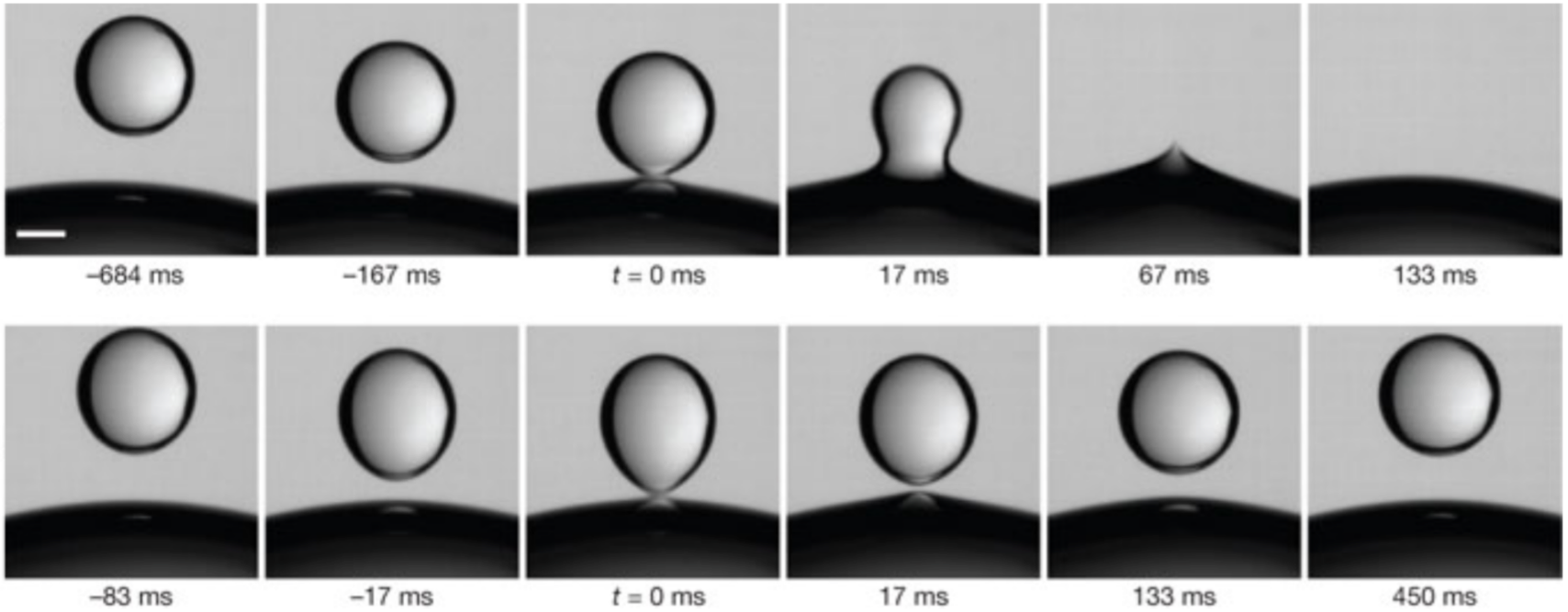
Angenent-Chopp-Ilmanen, Ilmanen-White (90s):



Conical singularities $\rightarrow$ Nonuniqueness!

General case - Much less understood!

Coalescence and bouncing of water in oil:



Ristenpart et. al. (Nature '09), Helmensdorfer-Topping (EPL '13)

The main conjectures

Mean-Convex Neighborhood Conjecture

If M_t has a neck-singularity at (x_0, t_0), then there exists $\varepsilon = \varepsilon(x_0, t_0) > 0$ such that $M_t \cap B_\varepsilon(x_0)$ is mean-convex for $|t - t_0| < \varepsilon$.

The main conjectures

Uniqueness Conjecture

Mean curvature flow through neck-singularities is unique.

The main conjectures

Sphere Conjecture

Mean curvature flow of embedded 2-spheres is well posed.

Note: MCF of 2-spheres is the cousin of Ricci flow on 4-spheres.

The main conjectures

Multiplicity 1 Conjecture

All tangent flows have multiplicity 1.

Background

$\mathcal{M} = \{M_t\}_{t \geq 0}$ mean curvature flow.

Given a point $X_0 = (x_0, t_0)$ and $\lambda_i \to \infty$ consider tangent-flow at X_0:

$$\hat{\mathcal{M}}_{X_0} := \lim_{i \to \infty} \mathcal{D}_{\lambda_i}(\mathcal{M} - X_0)$$

Tangent-flows exist and are self-similarly shrinking (by the compactness theorem for Brakke flows and Huisken's monotonicity formula).

Neck-singularity: $\hat{\mathcal{M}}_{X_0} = \{S^1(\sqrt{2|t|}) \times \mathbb{R}\}_{t<0}$

The main theorems

Mean-Convex Neighborhood Theorem (Choi-H-Hershkovits)

If M_t has a neck-singularity at (x_0, t_0), then there exists $\varepsilon = \varepsilon(x_0, t_0) > 0$ such that $M_t \cap B_\varepsilon(x_0)$ is mean-convex for $|t - t_0| < \varepsilon$.

The main theorems

Uniqueness Theorem (Choi-H-Hershkovits)

Mean curvature flow through neck-singularities is unique.

The main theorems

Sphere Theorem* (Choi-H-Hershkovits)

Mean curvature flow of embedded 2-spheres is well-posed.

*assuming the multiplicity 1 conjecture.

Strategy

Mean-Convex Neighborhood Conjecture

$\Downarrow$ (Hershkovits-White)

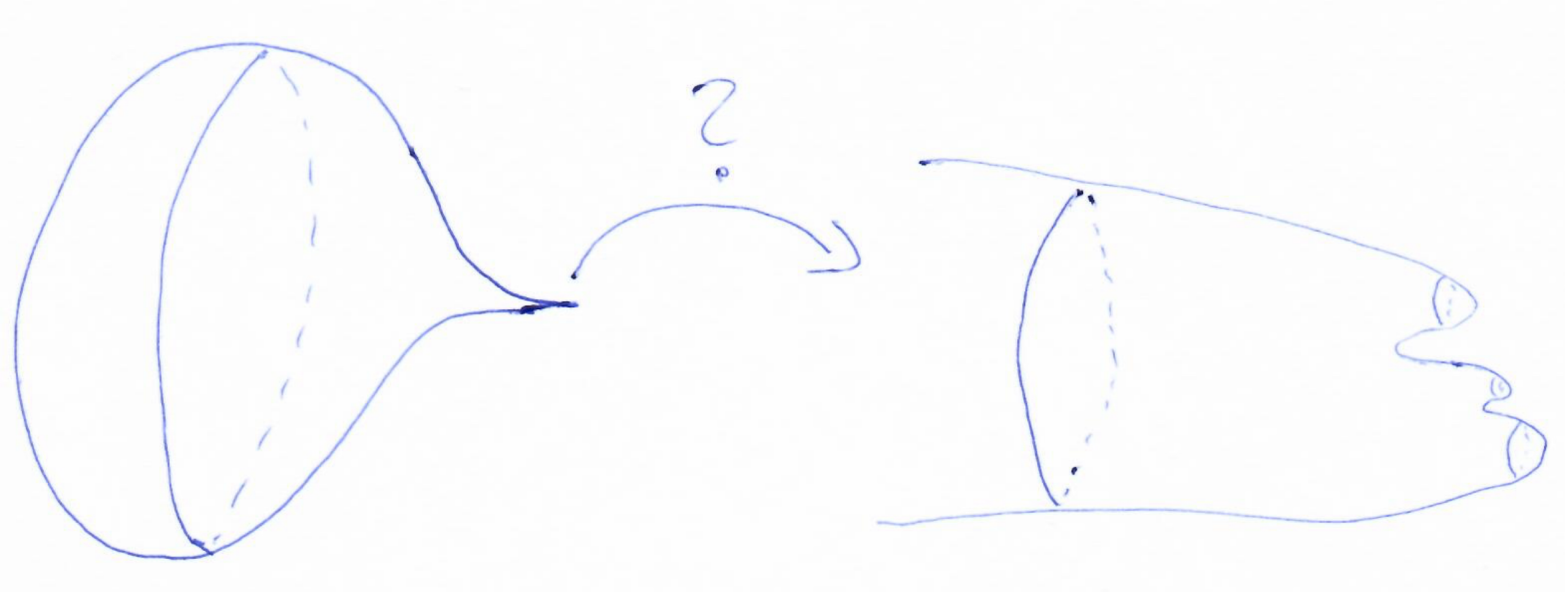
Uniqueness Conjecture

$\Downarrow^*$ (Brendle)

Sphere Conjecture

The main enemy

Potential scenario of a degenerate neck-pinch with a non-convex cap:



Limit flows

tangent-flow $\hat{\mathcal{M}} = \lim_{i\to\infty} \mathcal{D}_{\lambda_i}(\mathcal{M} - X_0)$ is a self-similarly shrinking cylinder by assumption.

more generally, for any $X_i \to X_0$, and any suitable $\lambda_i \to \infty$ we can consider the limit-flow: $\mathcal{M}^\infty = \lim_{i\to\infty} \mathcal{D}_{\lambda_i}(\mathcal{M} - X_i)$

Ancient asymptotically cylindrical flows

$\mathcal{M}^\infty$ is an ancient asymptotically cylindrical flow:

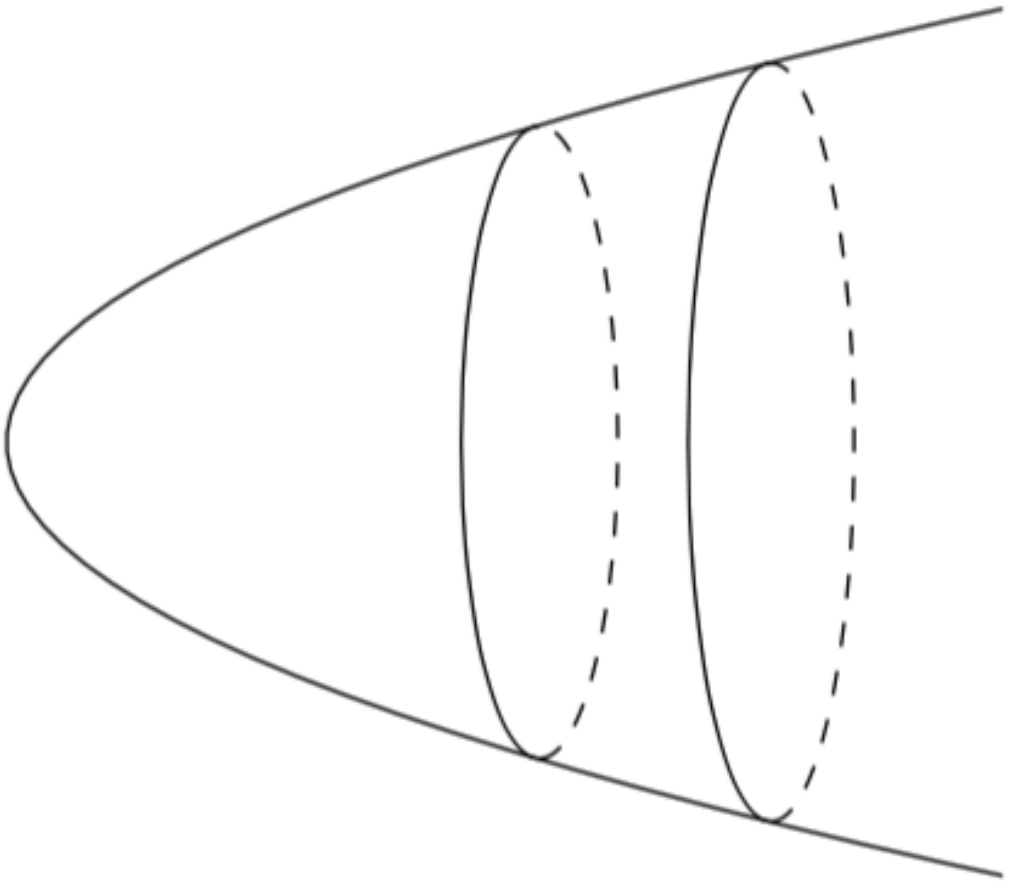
- defined for $t \in (-\infty, T_E)$
- blowdown: $\check{\mathcal{M}} = \lim_{\lambda_i \to 0} \mathcal{D}_{\lambda_i} \mathcal{M}^\infty = \{S^1(\sqrt{2|t|}) \times \mathbb{R}\}_{t<0}$
- unit-regular, cyclic, integral Brakke flow

Examples of ancient asymptotically cylindrical flows

Round shrinking cylinder

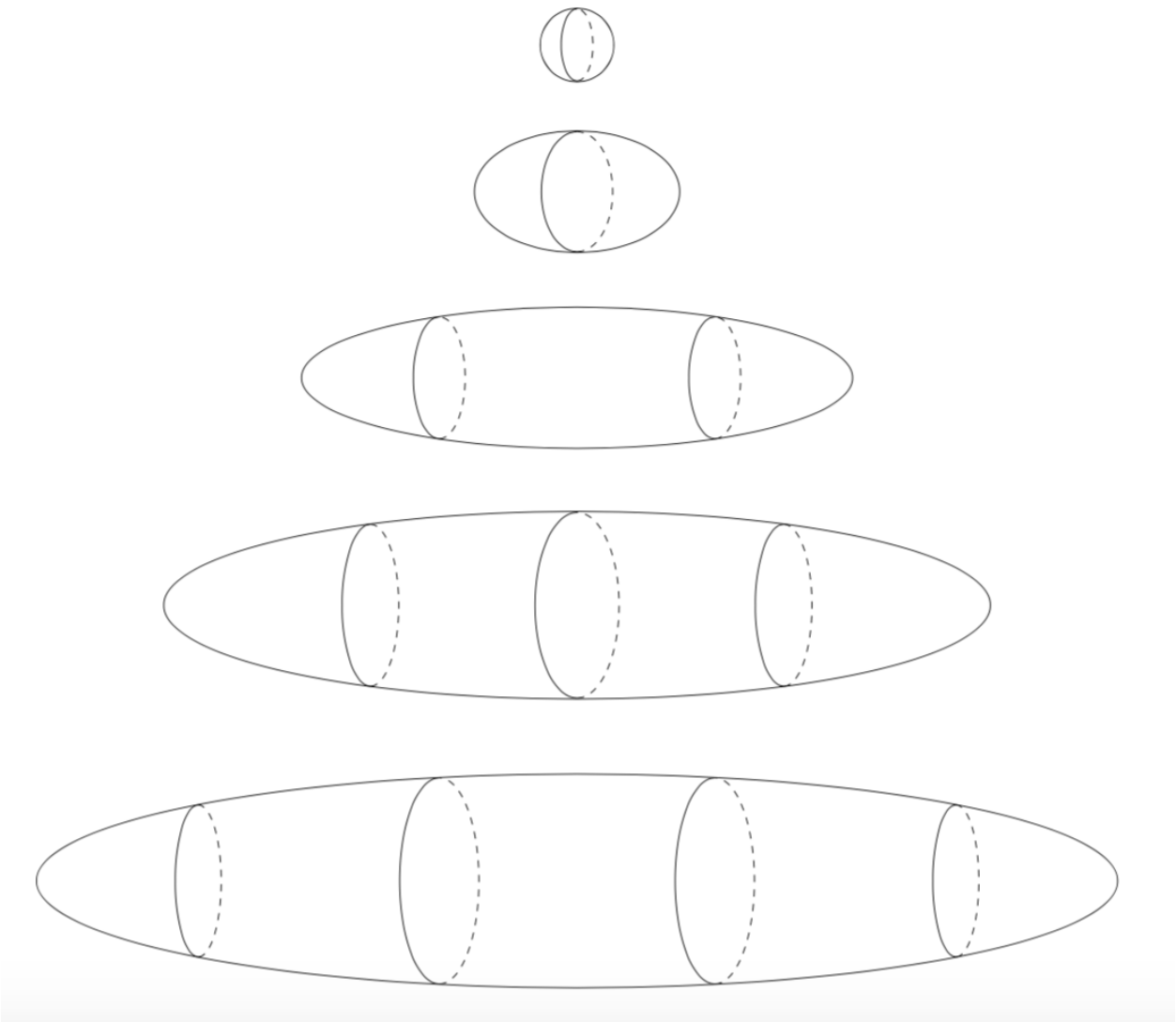
Examples of ancient asymptotically cylindrical flows

Translating bowl



Examples of ancient asymptotically cylindrical flows

Ancient ovals

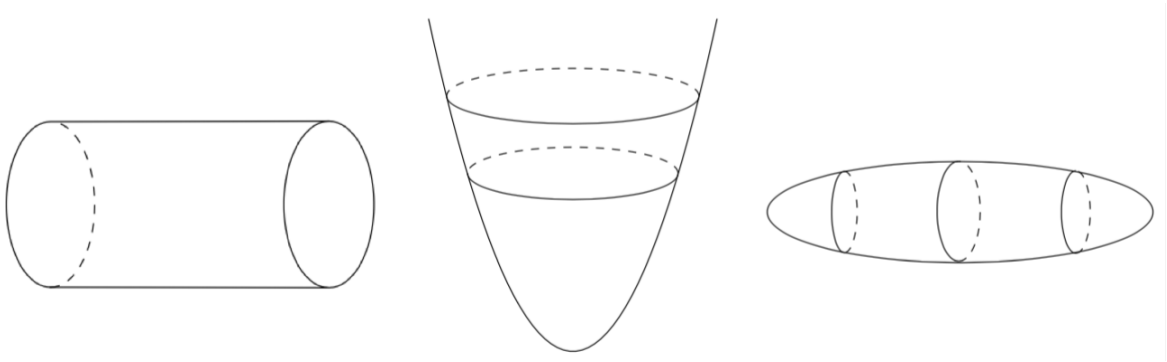


Classification of ancient asymptotically cylindrical flows

Classification Theorem (Choi-H-Hershkovits)

Any ancient asymptotically cylindrical flow is one of the following:

- *round shrinking cylinder*
- *translating bowl*
- *ancient oval*



In particular, any ancient asymptotically cylindrical flow is convex. Hence, the mean-convex neighborhood theorem follows.

Important prior classification results

- Wang (Annals '11): convex, self-similarly translating, entire graph $\Rightarrow$ bowl.
- H (Geom&Top '15): self-similarly translating, noncollapsed $\Rightarrow$ bowl.
- Bernstein-Wang (Duke '17): low entropy, self-similarly shrinking $\Rightarrow$ sphere or cylinder.
- Brendle-Choi (Inventiones '19): ancient, noncollapsed, noncompact $\Rightarrow$ cylinder or bowl.
- Angenent-Daskalopoulos-Sesum (Annals '20): ancient, noncollapsed, compact $\Rightarrow$ oval.

Recall that ancient noncollapsed flows are always convex by the H-Kleiner convexity estimate from Lecture 3.

Classification of ancient asymptotically cylindrical flows

Classification Theorem (Choi-H-Hershkovits)

Any ancient asymptotically cylindrical flow is one of the following:

- *round shrinking cylinder*
- *translating bowl*
- *ancient oval*

We assume neither self-similarity nor convexity! This is of crucial importance for the proof of the mean-convex neighborhood conjecture.

Idea of the proof

$\mathcal{M}$ ancient asymptotically cylindrical flow $\neq$ cylinder

to show: $\mathcal{M}$ = bowl or oval

Set up fine-neck analysis:

Given $X_0 = (x_0, t_0)$ consider normalized flow $\bar{M}_\tau^{X_0} = e^{\tau/2}(M_{t_0 - e^{-\tau}} - x_0)$.

Write $\bar{M}_\tau = \mathrm{graph}(u(\tau))$ locally. The analysis is governed by the Ornstein-Uhlenbeck operator $\mathcal{L} = \partial_z^2 - \frac{1}{2} z \partial_z + \frac{1}{2}\partial_\theta^2 + 1$

unstable-mode: $1, \sin\theta, \cos\theta, z$ neutral-mode: $z\sin\theta, z\cos\theta, z^2 - 2$

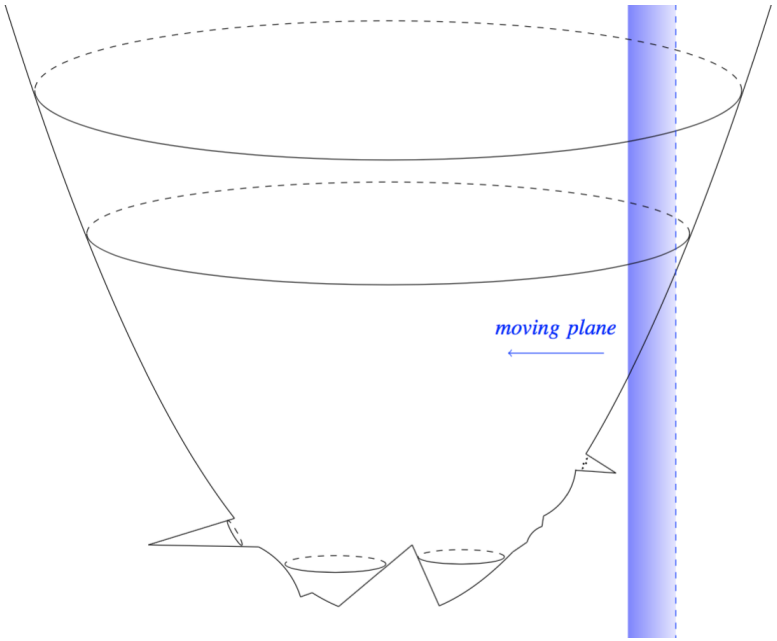
Merle-Zaag ODE lemma $\Rightarrow$ for $\tau \to -\infty$ either the unstable-mode is dominant or the neutral-mode is dominant.

Idea of the proof

Analysis in the unstable-mode:

Fine-neck theorem: There exists $a = a(\mathcal{M}) \neq 0$ independent of the center X_0, such that $u^{X_0}(z, \theta, \tau) = aze^{\tau/2} + o(e^{\tau/2}) \quad \forall \tau \ll \log Z(X_0)$.

Cap-size control theorem: There exists $C < \infty$ such that for every X_0 outside of a ball of radius C lies on a fine-neck.



Idea of the proof

Analysis in the unstable-mode (cont'd):

By the Brendle-Choi neck-improvement theorem, the solution becomes very symmetric as $r \to \infty$.

Moving plane method $\Rightarrow$ smooth and rotationally symmetric $\Rightarrow$ bowl.

Idea of the proof

Analysis in the neutral-mode:

$z^2 - 2$ dominant, analysis of coefficients $\Rightarrow$ inwards quadratic, compact near the tips, by classification in unstable-mode case we see bowls

maximum principle $\Rightarrow$ mean-convex, noncollapsed

general theory (e.g. H-Kleiner) $\Rightarrow$ smooth and convex

Angenent-Daskalopoulos-Sesum $\Rightarrow$ oval

Open problems

- Multiplicity 1 conjecture: All blowup limits have multiplicity-one.
- No cylinder conjecture: The only complete embedded shrinker with a cylindrical end is the round cylinder.
- Uniqueness of tangent-flows: Tangent flows are independent of the choice of sequence $\lambda_i \to \infty$.
- Bounded diameter conjecture: The intrinsic diameter stays uniformly bounded as one approaches the first singular time.
- Genericity conjecture: For generic initial data all singularities are of neck-type or spherical-type.

Open problems

- Partial regularity conjecture: $\dim_P \mathcal{S} \leq 1$.
- Isolation conjecture: All singularities are isolated unless an entire tube shrinks to a closed curve.
- Finiteness conjecture: There are only finitely many singular times.
- Self-similarity problem: Are blowup limits always self-similar?
- Shrinking tubes: Which closed curves can arise as singular set?

Open problems

- Sphere conjecture: Mean curvature flow of embedded 2-spheres is well-posed.
- Is there a MCF proof of the Smale conjecture?
- Lusternik-Schnirelman conjecture: S^3 with arbitrary Riemannian metric g contains at least 4 embedded minimal 2-spheres.
- Is there a selection principle for flowing out of conical singularities?
- Develop a theory of weak solutions for the flow of immersed surfaces.