Monotonicity formula & ε-regularity

Recall, if M_t moves by MCF, then

$$\frac{d}{dt}\,\mathrm{Area}(M_t) = -\int_{M_t} H^2\, d\mu$$

However, since $\mathrm{Area}(\lambda M) = \lambda^2\,\mathrm{Area}(M)$, this is not super useful for $\lambda \to \infty$.

$$\mathrm{Area}(M) = \int_M 1\, d\mu$$

$$\text{Gaussian-Area}(M_t) = \int_{M_t} \rho_{X_0}(x,t)\, d\mu(x)$$

where

$$\rho_{X_0}(x,t) = \frac{1}{4\pi(t_0-t)}\, e^{-\frac{|x-x_0|^2}{4(t_0-t)}}$$

$X_0 = (x_0, t_0)$ $\qquad (t < t_0)$

Localizes at scale $|x-x_0|^2 \lesssim |t_0-t|$

Huisken's monotonicity formula

$$\boxed{\frac{d}{dt}\int_{M_t} \rho_{x_0}\, d\mu_t = -\int_{M_t} \left|\vec{H} - \frac{(x-x_0)^\perp}{2(t-t_0)}\right|^2 \rho_{x_0}\, d\mu_t}$$

<u>Exer</u> (scaling invariance)

Let $x' = \lambda(x-x_0)$, $t' = \lambda^2(t-t_0)$, and consider the rescaled flow

$$M^\lambda_{t'} = \lambda\left(M_{t_0+\lambda^{-2}t'} - x_0\right).$$

Prove that

$$\int_{M_t} \rho_{x_0}(x,t)\, d\mu_t(x) = \int_{M^\lambda_{t'}} \rho_0(x',t')\, d\mu_{t'}(x)$$

<u>Exer</u> (shrinkers) Let $\{M_t \subset \mathbb{R}^3\}_{t\in(-\infty,0)}$

be an ancient solution of the MCF.

Prove that:

$$\vec{H} - \frac{x^\perp}{2t} = 0 \quad (\forall t < 0)$$

$$\Longleftrightarrow \quad M_t = \sqrt{-t}\, M_{-1} \quad (\forall t < 0)$$

<u>Proof</u> Wlog $X_0 = (0,0)$. Write $\rho = \rho_0$.

<u>Claim</u> $\left(\frac{d}{dt} + \Delta_{M_t} - H^2\right)\rho = -\left|\vec{H} - \frac{x^\perp}{2t}\right|^2 \rho.$

<u>Claim $\Rightarrow$ Thm</u> $\quad \frac{d}{dt} d\mu_t = -H^2 d\mu_t$

$$\frac{d}{dt}\int \rho\, d\mu_t = \int \left(\frac{d}{dt}\rho - H^2\rho\right) d\mu_t$$

$$= -\int \left|\vec{H} - \frac{x^\perp}{2t}\right|^2 \rho\, d\mu_t \qquad \boxed{//}$$

Claim & $\int_M \Delta\rho = 0$

Proof of Claim

$$\underbrace{\nabla^{M_t}\rho}_{\text{tangential gradient}} = \underbrace{D\rho}_{\text{gradient in } \mathbb{R}^3} - (D\rho\cdot\vec{\nu})\,\vec{\nu}$$

$$\Rightarrow \Delta_{M_t}\rho = \operatorname{div}_{M_t}(\nabla^{M_t}\rho)$$

$$= \operatorname{div}_{M_t}(D\rho) + \vec{H}\cdot D\rho$$

$$\operatorname{div}(\vec{\nu}) = -H$$

•) $\frac{d}{dt}\rho = \partial_t\rho + \vec{H}\cdot D\rho$ along MCF.

$$\Rightarrow \left(\frac{d}{dt} + \Delta_{M_t}\right)\rho = \partial_t\rho + \operatorname{div}_{M_t}(D\rho) + 2\vec{H}\cdot D\rho$$

$$= \underbrace{\partial_t\rho + \operatorname{div}_{M_t}(D\rho) + \frac{|\nabla^{\perp}\rho|^2}{\rho}}_{=0 \text{ (Exer)}} - \left|\vec{H} - \frac{\nabla^{\perp}\rho}{\rho}\right|^2\rho + H^2\rho$$

$\boxed{///}$

$\mathcal{M} = \{M_t\}$ MCF

$$\Theta(\mathcal{M}, X_0, r) := \int_{M_{t_0 - r^2}} \rho_{X_0} \, d\mu$$

Gaussian area at time $t = t_0 - r^2$.

<u>Note:</u> $\Theta(\mathcal{M}, X_0, r) = 1 \ \forall r \iff$

$\mathcal{M}$ is a multiplicity-one plane containing X_0.

<u>Thm</u> (ε-regularity)

$\exists\, \varepsilon > 0,\ C < \infty$ universal:

If $\mathcal{M}$ is a smooth MCF

with $\sup_{X \in P(X_0, r)} \Theta(\mathcal{M}, X, r) < 1 + \varepsilon$,

then $\sup_{P(X_0, r/2)} |A| \leq C r^{-1}$.

<u>Proof</u> If not, $\exists\, \mathcal{M}^j$ st

$$\sup_{X \in P(0,1)} \Theta(\mathcal{M}^j, X, 1) < 1 + j^{-1},$$

but $\exists\, X_j \in P(0, 1/2) : |A|(X_j) > j$.

By point-selection, $\exists\, Y_j \in P(0, 3/4)$ with $Q_j := |A|(Y_j) > j$ such that

$$\sup_{P(Y_j, \frac{j}{10Q_j})} |A| \leq 2Q_j .$$

Now let $\hat{\mathcal{M}}^j$ be the sequence of flows that is obtained from $\mathcal{M}^j$ by shifting Y_j to the origin and parabolically rescaling by Q_j.

Then $\hat{M}^j$ satisfies

$$|A|(0) = 1$$

$$\sup_{P(0,\frac{j}{10})} |A| \leq 2$$

$\Rightarrow$ $\hat{M}^{j'}$ converges smoothly to a nonflat limit.

However $\Theta(\hat{M}^j, 0, Q_j) < 1 + j^{-1}$

Monotonicity formula $\Rightarrow$ $\hat{M}^j \to$ flat plane

$\lightning$ $\boxed{//}$

$$\partial_t u = \Delta u$$

$$\frac{d}{dt} \int u \rho$$

need $\partial_t \rho = -\Delta \rho$ to make this monotone!