Monotonicity formula & $\varepsilon$-regularity

Recall, if $M_t$ moves by MCF, then

$$\frac{d}{dt} \text{Area}(M_t) = -\int_{M_t} H^2 \, d\mu$$

However, since $\text{Area}(\lambda M) = \lambda^2 \text{Area}(M)$, this is not super useful for $\lambda \to \infty$.

$$\text{Area}(M) = \int_M 1 \, d\mu$$

Gaussian-Area $(M_t) = \int_{M_t} \rho_{X_0}(x,t) \, d\mu(x)$

where

$$\rho_{X_0}(x,t) = \frac{1}{4\pi(t_0-t)} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}$$

$X_0 = (x_0, t_0)$

Locallizes at scale $|x-x_0|^2 \leq |t_0-t|$
Huiskens's monotonicity formula

\[
\frac{d}{dt} \int_{\mathcal{M}_t} p_{x_0} \, d\mu_t = - \int_{\mathcal{M}_t} \left| \nabla - \frac{(x-x_0)\perp}{2(t-t_0)} \right|^2 \rho_{x_0} \, d\mu_t
\]

**Exer (scaling invariance)**

Let \( x' = \lambda(x-x_0) \), \( t' = \lambda^2(t-t_0) \), and consider the rescaled flow

\[
\mathcal{M}_{t'}^\lambda = \lambda (\mathcal{M}_{t_0} + x^{-2}t' - x_0) .
\]

Prove that

\[
\int_{\mathcal{M}_t} p_{x_0}(x,t) \, d\mu_t(x) = \int_{\mathcal{M}_{t'}^\lambda} p_0(x',t') \, d\mu_{t'}(x)
\]

**Exer (shrinkers)** Let \( \{ \mathcal{M}_t \subset \mathbb{R}^3 \} \)
be an ancient solution of the MCF.

Prove that:

\[ \vec{H} - \frac{x^1}{2t} = 0 \quad (\forall t < 0) \]

\[ \iff M_t = \sqrt{-t} M_{-1} \quad (\forall t < 0) \]

Proof. We set \( X_0 = (0,0) \). Write \( \rho = \rho_0 \).

Claim:

\[ \frac{d}{dt} \left( \frac{1}{2t} + \Delta M_t - H^2 \right) \rho = -\sqrt{H^2 - \frac{x^1}{2t}}^2 \rho. \]

Claim \( \Rightarrow \) Thm

\[ \frac{d}{dt} \int \rho \, d\mu_t = -H^2 \rho \, d\mu_t \]

\[ \frac{d}{dt} \int \rho \, d\mu_t = \int \left( \frac{d}{dt} \rho - H^2 \rho \right) \, d\mu_t \]

\[ = -\int \sqrt{H^2 - \frac{x^1}{2t}}^2 \rho \, d\mu_t \]

\( \forall \) Claim & \( \int \Delta \rho = 0 \)

\[ \Box \]
Proof of Claim

\[ \nabla_{H_t} \rho = D\rho - (D\rho \cdot \nabla_{H_t} \rho) \nabla_{H_t} \rho \]

tangential gradient in \( \mathbb{R}^3 \)

gradient

\[ \Rightarrow \Delta_{H_t} \rho = \text{div}_{H_t} (\nabla_{H_t} \rho) \]

\[ = \text{div}_{H_t} (D\rho) + \tilde{H} \cdot D\rho \]

\[ \text{div} (\nabla_{H_t} \rho) = -\tilde{H} \]

\[ \cdot \]

\[ \frac{d}{dt} \rho = \partial_t \rho + \tilde{H} \cdot D\rho \quad \text{along MCF} \]

\[ \Rightarrow \left( \frac{d}{dt} + \Delta_{H_t} \right) \rho = \partial_t \rho + \text{div}_{H_t} (D\rho) + 2\tilde{H} \cdot D\rho \]

\[ = \partial_t \rho + \text{div}_{H_t} (D\rho) + \frac{|D\rho|^2}{\rho} - \tilde{H} - \frac{|\nabla_{H_t} \rho|^2}{\rho} + \tilde{H}^2 \rho \]

\[ = 0 \quad (\text{Exer}) \]
\[ M = \{ M_{t_0} \} \quad \text{MCF} \]

\[ \mathcal{H}(M, x_0, r) := \int_{M_{t_0 - r^2}} p_{x_0} \, d\mu \]

Gaussian area at time \( t = t_0 - r^2 \).

Note: \( \mathcal{H}(M, x_0, r) = 1 \) \( \forall r \leq \)

\( M \) is a multiplicity-one plane containing \( x_0 \).

Then (\( \varepsilon \)-regularity)

\[ \exists \varepsilon > 0, C < \infty \text{ universal:} \]

If \( M \) is a smooth MCF

with \( \sup \mathcal{H}(M, x, r) < 1 + \varepsilon \), \( x \in P(x_0, r) \)

then \( \sup_{P(x_0, r/2)} |A| \leq C r^{-1} \).
Proof: If not, \( \exists M \) s.t.

\[
\sup_{X \in \mathcal{P}(0, 1)} \Theta(\mathcal{H}^\dagger, X, 1) < 1 + j^{-1},
\]

but \( \exists X_j \in \mathcal{P}(0, 1/2) : |A|_1(X_j) > j \).

By point-selection, \( \exists Y_j \in \mathcal{P}(0, 3/4) \) with \( Q_j := |A|_1(Y_j) > j \) such that

\[
\sup_{P(Y_j, Y_j)} |A| \leq 2Q_j.
\]

Now let \( \hat{\mathcal{H}}^\dagger \) be the sequence of flows that is obtained from \( \mathcal{H}^\dagger \) by shifting \( Y_j \) to the origin and parabolically rescaling by \( Q_j \).
Then \( \hat{A} \) satisfies
\[
|A|_0(0) = 1
\]
\[
\sup_{P(0, t_0)} |A| \leq 2
\]
\[
\Rightarrow \hat{A} \text{ converges smoothly to a non-flat limit.}
\]

However \( \Theta（\hat{A}, 0, Q_j） < 1 + j^{-1} \)

Monotonicity formula \( \Rightarrow \hat{A} \rightarrow \text{flat plane} \)

\[
\underline{\partial_t u = \Delta u}
\]
\[
\frac{d}{dt} \sup \text{ need } \partial_t \rho = -\Delta \rho \text{ to make this monotone!}
\]