MOVING PLANE METHOD FOR VARIFOLDS AND APPLICATIONS

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Dedicated to the memory of Professor Louis Nirenberg

ABSTRACT. In this paper, we introduce a version of the moving plane method that applies to potentially quite singular hypersurfaces, generalizing the classical moving plane method for smooth hypersurfaces. Loosely speaking, our version for varifolds shows that smoothness and symmetry at infinity (respectively at the boundary) can be promoted to smoothness and symmetry in the interior. The key feature, in contrast with the classical formulation of the moving plane principle, is that smoothness is a conclusion rather than an assumption.

We implement our moving plane method in the setting of compact varifolds with smooth boundary and in the setting of complete varifolds. A key ingredient is a Hopf lemma for stationary and CMC-varifolds. Our Hopf lemma provides a new tool to establish smoothness of varifolds, and works in arbitrary dimensions and without any stability assumptions. As applications of our new moving plane method, we prove varifold uniqueness results for the catenoid, spherical caps, and Delaunay surfaces that are inspired by classical uniqueness results by Schoen, Alexandrov, Meeks and Korevaar-Kusner-Solomon. We also prove a varifold version of Alexandrov's Theorem for compact CMC-varifolds in hyperbolic space.

1. INTRODUCTION

The moving plane method is a fundamental tool to establish symmetry in geometry and partial differential equations. This method was pioneered by Alexandrov [Ale62], Serrin [Ser71], Gidas-Ni-Nirenberg [GNN79], Berestycki-Nirenberg [BN91], and Schoen [Sch83]. Since then, the method has been applied frequently in the literature. For surveys and comprehensive references see e.g. Brezis [Bre99] and Ciraolo-Roncoroni [CR18]. Loosely speaking, the method gives rise to the following general principle:

Principle 1.1 (Classical moving plane principle)**.** *Symmetry at infinity (or at the boundary) can be promoted to symmetry in the interior.*

In our recent joint work with Choi, we discovered a new variant of the moving plane principle, which can be loosely speaking stated as follows:

Principle 1.2 (New moving plane principle [CHHW19])**.** *Smoothness and symmetry at infinity (or at the boundary) can be promoted to smoothness and symmetry in the interior.*

The gist of this principle is illustrated in Figure 1. The key feature, in contrast with the classical moving plane principle, is that smoothness is a conclusion, not an assumption. This was of crucial importance in our recent proof of the canonical neighborhood conjecture for mean curvature flow through neck-singularities in arbitrary dimensions [CHHW19]. There, the blowup limits that we encountered near
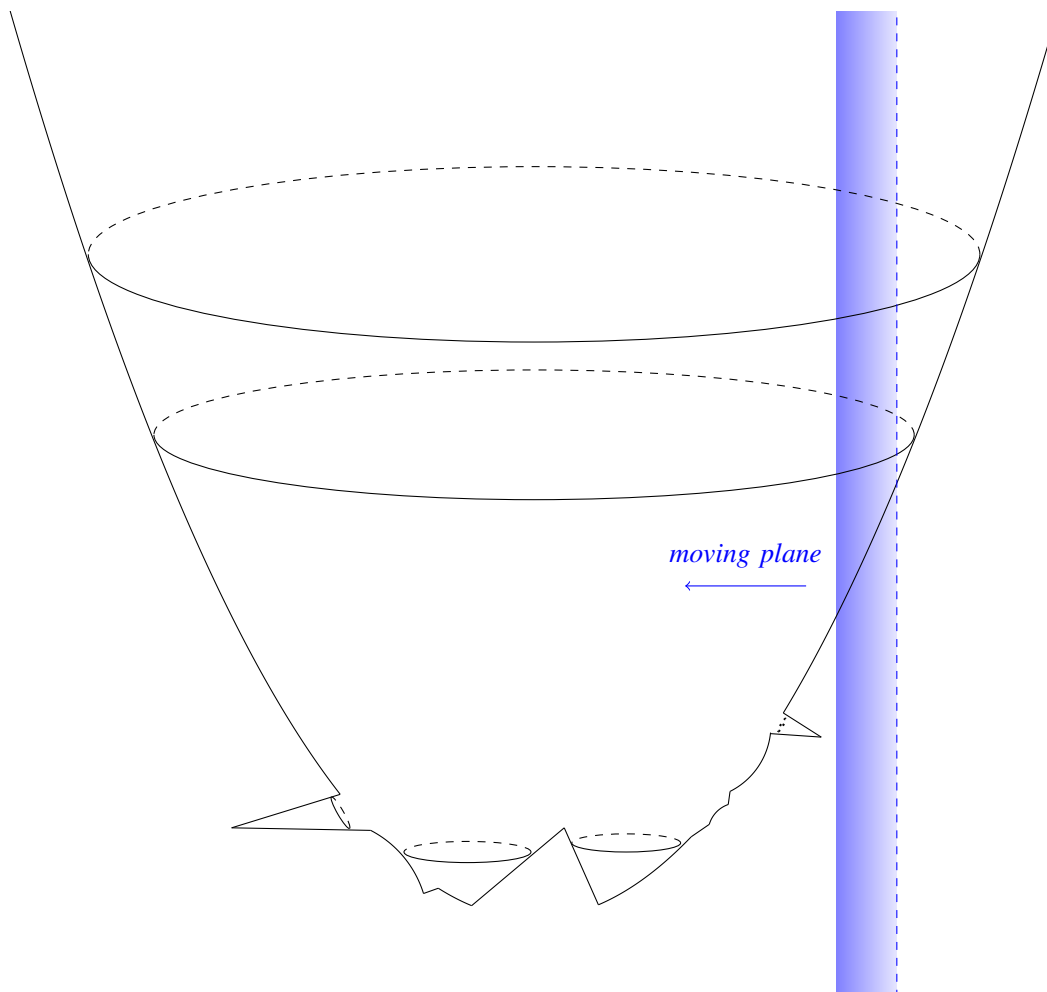
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FIGURE 1. The cap region could a priori be quite singular. Whenever the moving plane reaches a point (away from the axis of symmetry) then this point must be smooth.

a neck-singularity were potentially quite singular objects (integral Brakke flows satisfying a few technical conditions), and in order to classify them we had to establish smoothness and symmetry in tandem. Unfortunately, the parabolic (rather than elliptic) nature of [CHHW19], together with the many other ingredients of that paper, may obscure the simplicity and generality of Principle 1.2.

In the present paper, which we hope is accessible to a larger class of readers, we develop the method in the elliptic setting of potentially singular hypersurfaces. Specifically, we implement the method to prove uniqueness results for stationary varifolds and CMC-varifolds, analogous to classical uniqueness results for smooth hypersurfaces by Schoen [Sch83], Alexandrov [Ale62], Meeks [Mee88] and Korevaar-Kusner-Solomon [KKS89]. These uniqueness results will be discussed in Section 1.1 and Section 1.2, respectively. A description of our new method itself, including in particular a Hopf lemma for varifolds – which seems to be of independent interest – will be given in Section 1.3.

We recently learned that Bernstein-Maggi [BM20] have developed a very interesting variant of the moving plane method that allows them to prove symmetry of certain singular Plateau surfaces, and seems rather different from our method which establishes smoothness.

1.1. **Uniqueness results for stationary varifolds.** In this section, we describe our uniqueness results for potentially singular minimal hypersurfaces in $\mathbb{R}^{n+1}$. These potentially singular objects are described most conveniently as stationary integral n-varifolds, as introduced by Almgren and Allard [Alm65, All72]. An n-varifold is a measure-theoretic generalization of an n-dimensional surface. Integral means that at almost every point one can find a tangent plane of integer multiplicity, and stationary means that the varifold is a critical point of the n-dimensional area functional (see Section 2 for precise definitions and notation).

The moving plane method for noncompact hypersurfaces was pioneered by Schoen [Sch83], who proved uniqueness of the catenoid among smooth minimal hypersurfaces that are asymptotic to two planes. Our first main theorem extends Schoen's uniqueness result to the setting of varifolds:

Theorem 1.3 (Uniqueness of the catenoid). *Suppose M is stationary complete integral n-varifold in $\mathbb{R}^{n+1}$ such that*

    (i) some tangent cone at infinity is a multiplicity-2 plane,
    (ii) M has at least two ends,[1]
    (iii) M has no triple junctions.

Then M is a smooth hypersurface of revolution, and thus either a pair of parallel planes or an n-dimensional catenoid.

The main novelty is that smoothness is a conclusion of our theorem rather than an assumption. In the study of minimal hypersurfaces, one often passes to weak limits, which could be potentially quite singular. Theorem 1.3 does not assume smoothness a priori, and thus can be used to analyze such limits.

In the same paper [Sch83], Schoen also proved symmetry for smooth compact minimal hypersurfaces with rotationally symmetric boundary. We extend this result as well:

Theorem 1.4 (smoothness and symmetry for stationary varifolds with smooth boundary). *Suppose that $\Gamma = \Gamma_1 \cup \Gamma_2$ is the union of two smooth, closed, strictly convex $(n-1)$-dimensional manifolds lying in parallel hyperplanes Q_1 and Q_2 in $\mathbb{R}^{n+1}$. Suppose M is a compact connected integral n-varifold in $\mathbb{R}^{n+1}$ such that*

    (i) M is stationary in $\mathbb{R}^{n+1}\backslash\Gamma$,
    (ii) The density of M at each point of Γ is $1/2$,
    (iii) M has no triple junction points.

If P is a hyperplane of symmetry of the boundary Γ, then P is a plane of symmetry of M. Moreover, the portion of M on each side of P is a smooth graph over a region in P.

In particular, if Γ is rotationally symmetric, then M is a piece of an n-dimensional catenoid.

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[1]By the convex hull property, M has at least two ends if and only if there is a ball B such that $M\backslash B$ is not connected.

An interesting feature of the theorem is that it gives smoothness (away from the plane of symmetry) even if the boundary is symmetric only with respect to a single plane P, and not necessarily rotationally symmetric. Combining Theorem 1.4 with some results from [Whi19] we also obtain:

Corollary 1.5. *If the hypotheses* (i), (ii) *and* (iii) *are replaced by the hypotheses*

(i′) $\int \mathrm{Div}_M X \, d\mu_M \leqslant \int_\Gamma |X| \, d\mathcal{H}^{m-1}$ *for every C^1 vectorfield X on $\mathbb{R}^{n+1}$, and*

(ii′) *The boundary of the mod 2 flat chain associated to M is $[\Gamma]$,*

then the conclusion of Theorem 1.4 holds.

For readers not familiar with mod 2 flat chains, Hypothesis (ii′) is equivalent to the statement: If C is a smooth simple closed curve in $\mathbb{R}^{n+1}$, then for almost every $v \in \mathbb{R}^{n+1}$, the number of intersections of $(C + v) \cap M$ (counting multiplicity) is equal to the mod 2 linking number of $C + v$ and Γ.

1.2. **Uniqueness results for CMC-varifolds.** In this section, we consider possibly singular hypersurfaces with constant nonzero mean curvature. Variationally, such CMC-varifolds arise as critical points of the n-dimensional area functional subject to the constraint that the enclosed volume is kept fixed. Our results in this section require the following regularity assumption.[2]

Definition 1.6 (tameness)**.** We say that M is a *tame CMC-varifold* in $\mathbb{R}^{n+1}$ provided there is a smooth $(n-1)$-manifold Γ (called the *boundary* of M) such that

(i) M is an integral n-varifold in $\mathbb{R}^{n+1}$,
(ii) M is CMC in $\mathbb{R}^{n+1} \backslash \Gamma$,
(iii) At each point in $M \backslash \Gamma$, each tangent cone is smooth with multiplicity one away from a set of $(n-1)$-dimensional Hausdorff measure 0, and
(iv) At each point of Γ, the density is $1/2$.

Recall that the moving plane method was introduced by Alexandrov [Ale62], who proved that every compact connected smooth embedded CMC-hypersurface must be a round sphere. A related classical result of Meeks [Mee88] states that there is no noncompact smooth embedded CMC-surface in $\mathbb{R}^3$ with a single end. Inspired by these results, we have the following uniqueness theorem for spherical caps:

Theorem 1.7 (Uniqueness of CMC spherical caps)**.** *Let M be a compact connected tame CMC-varifold in $\mathbb{R}^{n+1}$ whose boundary is an $(n-1)$-sphere in an n-plane P. Suppose that M lies on one side of P, and meets P transversally. Then M is a portion of a round n-sphere.*

We remark that for the Alexandrov theorem itself, there is a beautiful recent result by Delgadino-Maggi [DM19], which establishes uniqueness among compact connected sets of finite perimeter without tameness assumption. Their proof relies on a clever use of the Heintze-Karcher inequality and generalizes an earlier argument by Montiel-Ros [MR91].

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[2]In the stationary case, we will see that tameness can be deduced, using monotonicity, from the assumptions of Theorem 1.3 and Theorem 1.4, respectively. However, in the CMC case, tameness does not follow readily from global assumptions.

The main advantage of our own method is that it seems to have a much wider scope. In particular, we can deal with boundaries and with situations with much less symmetries. Moreover, our method also applies to certain other ambient manifolds as illustrated by the following theorem:

Theorem 1.8. *Let M be a compact connected tame CMC-varifold in an $(n + 1)$-dimensional Riemannian manifold N. If N is Euclidean space or hyperbolic space, then M is a geodesic sphere. If N is (n-dimensional hyperbolic space)$\times\mathbb{R}$, then M is smooth and rotationally invariant about an axis $\{p\} \times \mathbb{R}$.*

Regarding CMC-varifolds with two boundary components, the following is inspired by the classical uniqueness theorem for Delaunay strips by Korevaar-Kusner-Solomon [KKS89]:

Theorem 1.9 (Uniqueness of Delaunay strips). *Let M be a compact connected tame CMC-varifold in $\mathbb{R}^{n+1}$ with boundary consisting of a pair of $(n - 1)$-spheres that lie in parallel n-planes and that have the same axis of rotational symmetry. Suppose that M is contained in the slab between the n-planes, and meets these planes transversally. Then M is a smooth embedded hypersurface of revolution, and hence a piece of a Delaunay hypersurface.*

1.3. **Moving plane method for varifolds.** Recall that the classical moving plane method relies on the maximum principle and the Hopf Lemma in the smooth setting. To implement our new variant of the moving plane method, we need generalizations of the maximum principle and the Hopf lemma to the varifold setting.

To discuss our Hopf lemma for varifolds, denote by $\mathbb{H} \subset \mathbb{R}^{n+1}$ an open halfspace whose boundary n-plane contains the origin. Recall first that the classical Hopf lemma says that if u_1, u_2 are smooth solutions of a second order elliptic partial differential equation, defined in an open ball $B(0, r)$, such that $u_1(0) = u_2(0)$ and $u_1(x) < u_2(x)$ for all $x \in B(0, r) \cap \mathbb{H}$, then u_1 and u_2 have distinct normal derivatives at 0. The first guess regarding how to generalize this for varifolds would be to infer that the tangent cones at 0 must be distinct. But actually our conclusion is much better. In essence, we can use the fact that one varifold lies above the other one to *conclude* that 0 must be a smooth point. Specifically, we prove:

Theorem 1.10 (Hopf lemma for varifolds). *Let M_1 and M_2 be integral n-varifolds in $B(0, r) \subset \mathbb{R}^{n+1}$ that are stationary or that have the same constant mean curvature $h \neq 0$. If*

   (i) $0 \in M_1 \cap M_2$ is a tame interior point for both M_1 and M_2,
   *(ii) $\partial\mathbb{H}$ is **not** the tangent cone to either M_1 or M_2 at 0,*
   (iii) and reg $M_1 \cap \mathbb{H}$ and reg $M_2 \cap \mathbb{H}$ are disjoint,

then either

   (a) $h \neq 0$ and M_1 and M_2 are CMC-varifolds that curve oppositely at 0, or
   (b) M_1 and M_2 are smooth at 0, with distinct tangents.

The key feature of Theorem 1.10 is that smoothness is a conclusion and not an assumption. It thus provides a new tool to establish regularity of varifolds. Moreover, a quite unique advantage is that this

works without any dimensional restrictions and stability conditions (in contrast to other available methods where one usually assumes stability and $n < 7$). For our new moving plane method, the case of particular interest is when M_2 is the image of M_1 under reflection in the plane $\partial\mathbb{H}$.

Let us now explain some technical details: For a stationary or CMC integral n-varifold M, as in the previous subsection, a point $x \in M$ is called *tame* if each tangent cone of M at x is smooth with multiplicity one away from a set of $(n-1)$-dimensional Hausdorff measure zero. This tameness condition cannot be dropped, since the statement clearly fails for triple-junctions. Next, reg M simply denotes the set of regular points. Finally, in the CMC-case, we say that M_1 and M_2 *curve oppositely* at a tame point $x \in M_1 \cap M_2$ if M_1 and M_2 have a mutual tangent cone at x, but with the opposite orientation (see Definition 3.2). This scenario can often be ruled out in applications, e.g. when M_a arises as the boundary of some domain K_a.

Next, let us briefly discuss the maximum principle. Fortunately, nonsmooth versions of the maximum principle for hypersurfaces have been studied extensively in the literature (in stark contrast to the Hopf lemma). The sharpest result has been obtained by Wickramasekera [Wic14], building on earlier work by Simon [Sim87] and Ilmanen [Ilm96]. For our purpose, the following simple variant by Solomon and the last author, where one of the two varifolds is assumed to be smooth, is sufficient:

Theorem 1.11 (Maximum principle for varifolds [SW89])**.** *Let M_1 be a smooth minimal hypersurface defined in a ball $B(0, r)$ centered at $0 \in M_1$ and with small enough radius $r > 0$ such that M_1 separates $B(0, r)$ into two open connected components, U and U'. Let M_2 be a stationary integral n-varifold in $B(0, r)$, and assume that $0 \in M_2$ is a tame interior point. If*

$$(1) \qquad\qquad M_2 \subseteq U \cup M_1,$$

then 0 is a smooth point for M_2 and there exists some $\varepsilon > 0$ such that

$$(2) \qquad\qquad M_2 \cap B(0, \varepsilon) = M_1 \cap B(0, \varepsilon).$$

The same conclusion holds in the CMC-case, unless M_1 and M_2 curve oppositely at 0.

This is well-suited for our purpose, since smoothness of M_2 is not an assumption but a conclusion. Having discussed these tools, let us now describe the moving plane method for varifolds:

Description of the method. Suppose we would like to establish smoothness and symmetry of a stationary or CMC varifold M, with respect to a plane P. Denote by P_t the plane parallel to P at level t above P. The idea is then to push down the plane P_t, starting from a sufficiently high level, to establish smoothness and symmetry in tandem. More precisely, we want to show that for all $t > 0$ we simultaneously have:

(i) M can be reflected across P_t, *and*
(ii) the part of M that lies above P_t is smooth.

We first check that this is true for sufficiently large t, either by compactness or by suitable asymptotics. Suppose towards a contradiction we get stuck at some level $t > 0$. We then argue that this contradicts either the maximum principle for varifolds (Theorem 1.11) or the Hopf lemma for varifolds (Theorem

1.10). We can thus push the moving plane all the way to level 0, which establishes smoothness and symmetry.

This article is organized as follows. In Section 2, we summarize some preliminaries about varifolds. In Section 3, we prove our Hopf lemma for varifolds (Theorem 1.10). In Section 4, we prove our results for compact varifolds with boundary, specifically Theorem 1.4, Theorem 1.7 and Theorem 1.9. In Section 5, we address the noncompact case and prove uniqueness of the catenoid (Theorem 1.3). Finally, in Section 6, we explain how the method can be used in more general ambient manifolds, and prove Theorem 1.8.

2. Preliminaries and Notation

Standard references for varifolds are [All72, Sim83]. For a more gentle introduction we recommend [DL18]. Here, we briefly collect what we need for the present paper.

In the following, we denote by $\mathcal{H}^n$ the n-dimensional Hausdorff measure of subsets of $\mathbb{R}^{n+1}$. Recall that a set $M \subset \mathbb{R}^{n+1}$ is called *locally n-rectifiable* if it has locally finite $\mathcal{H}^n$-measure and is contained in a countable union of C^1-hypersurfaces, up to a set of $\mathcal{H}^n$-measure zero.

Definition 2.1 (integral varifold)**.** An *integral n-varifold in $\mathbb{R}^{n+1}$* is a pair $V = (M, \theta)$, where M is a locally n-rectifiable subset of $\mathbb{R}^{n+1}$ and $\theta : M \to \mathbb{Z}_{>0}$ is a locally integrable function.

We will often hide the multiplicity function θ in the notation, and simply talk about the varifold M.

Associated to V we have the measure $d\mu_M = \theta \, d\mathcal{H}^n \lfloor M$, which can be viewed as generalization of the n-dimensional area measure of smooth hypersurfaces. The *first variation* of V is given by[3]

$$(3) \qquad \delta V(X) = \int \operatorname{div}_{T_x M} X \, d\mu_M, \qquad \text{where } X \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}),$$

which generalizes the usual formula for the first variation of area of smooth hypersurfaces.

The varifold V is called *stationary*, if $\delta V = 0$. Equivalently, this means that the generalized mean curvature $\mathbf{H}$, defined via

$$(4) \qquad \delta V(X) = - \int X \cdot \mathbf{H} \, d\mu_M,$$

vanishes identically. A tame varifold is *CMC* if there exists a constant $h \neq 0$ such that $|\mathbf{H}| = h$.

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[3]Recall that any locally n-rectifiable $M \subset \mathbb{R}^{n+1}$ has an approximate tangent plane $T_x M$ at almost every $x \in M$.

By the monotonicity formula for stationary varifolds the function

$$(5) \qquad\qquad r \mapsto \frac{\mu_M(B(x,r))}{\omega_n r^n} \quad \text{is monotone,}$$

and constant only on cones. The CMC-case is similar: One only has to multiply by e^{hr}, which has negligible effect as $r \to 0$. In particular, every $x \in M$ has a well defined *density*

$$(6) \qquad\qquad \Theta(M,x) := \lim_{r \to 0} \frac{\mu_M(B(x,r))}{\omega_n r^n}.$$

By Allard's regularity theorem [All72], there exists some universal constant $\varepsilon = \varepsilon(n) > 0$ so that every point $x \in M$ with $\Theta(M,x) < 1 + \varepsilon$ is regular. In particular, if we decompose

$$(7) \qquad\qquad M = \operatorname{reg} M \cup \operatorname{sing} M,$$

then the regular part is open and the singular part is closed.

By Allard's compactness theorem [All72], any sequence $V_i = (M_i, \theta_i)$ of stationary or CMC integral n-varifolds with locally uniformly bounded measure has a subsequence that converges to a stationary or CMC integral n-varifold $V = (M, \theta)$. Here, convergence in the sense of varifolds means that[4]

$$(8) \qquad\qquad \int \varphi(x, T_x M_i)\, d\mu_{M_i}(x) \to \int \varphi(x, T_x M)\, d\mu_M(x)$$

for all compactly supported continuous functions φ on the Grassmannian $\operatorname{Gr}_n(\mathbb{R}^{n+1})$.

Given $V = (M, \theta)$, a point $x \in M$, and scales $\lambda_i \to \infty$, let $V_i = \lambda_i \cdot (V - x)$ be the sequence of varifolds that is obtained from V by shifting x to the origin and rescaling by λ_i. By the monotonicity formula and Allard's compactness theorem, we can always pass to a subsequential limit C, called a *tangent cone* at x. Tangent cones are always conical, i.e. $\lambda \cdot C = C$ for all $\lambda > 0$. In particular, if some tangent cone at x is a plane with multiplicity one (here, and in related situations, we also used that the multiplicity is simply a constant by the so-called constancy theorem [All72]), then x is regular by Allard's Regularity Theorem.

3. Hopf lemma for varifolds

The goal of this section is to prove the Hopf Lemma for varifolds (Theorem 1.10), which we restate here for convenience of the reader:

Theorem 3.1 (Hopf lemma for varifolds). *Let M_1 and M_2 be integral n-varifolds in $B(0,r) \subset \mathbb{R}^{n+1}$ that are stationary or that have the same constant mean curvature $h \neq 0$. If*

　(i) *$0 \in M_1 \cap M_2$ is a tame interior point for both M_1 and M_2,*
　(ii) *$\partial\mathbb{H}$ is **not** the tangent cone to either M_1 or M_2 at 0,*
　(iii) *and $\operatorname{reg} M_1 \cap \mathbb{H}$ and $\operatorname{reg} M_2 \cap \mathbb{H}$ are disjoint,*

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[4]In particular, this implies that μ_{M_i} converges to μ_M in the sense of measures. But the notion of convergence of varifolds also captures some important additional information about the convergence of the approximate tangent planes.

then either

    (a) $h \neq 0$ and M_1 and M_2 are CMC-varifolds that curve oppositely at 0, or

    (b) M_1 and M_2 are smooth at 0, with distinct tangents.

Here, the precise definition of curving oppositely, based on mutual tangent cones, is as follows:

Definition 3.2 (mutual tangent cones, curving oppositely). Let M_1 and M_2 be CMC-varifolds defined in an open ball $B(0, r)$, such that $0 \in M_1 \cap M_2$ is a tame interior point for both M_1 and M_2.

    (i) The collection of mutual tangent cones C at 0 is the collection of all pairs (Σ_1, Σ_2) such that there is a sequence $\lambda_j \to \infty$ for which $\lambda_j M_a$ converges to Σ_a for $a = 1, 2$.

    (ii) We say M_1 and M_2 curve oppositely at 0, if C has an element of the form (Σ, Σ) such that the orientations of $\operatorname{reg} \Sigma$ induced from the curvature vectors of $\operatorname{reg} M_1$ and $\operatorname{reg} M_2$ are inconsistent.

To prove Theorem 3.1, we first recall two Bernstein-type theorems for varifold shrinkers in a halfspace that were proved in our prior work [CHHW19, Sec. 3.2]. These theorems generalize important results for smooth two-dimensional surfaces by Brendle [Bre16].

Definition 3.3 (varifold shrinker, [CHHW19, Def. 3.5]). A *varifold shrinker* is an integral n-varifold Σ in $\mathbb{R}^{n+1}$ with finite entropy that is stationary with respect to the functional

$$(9) \qquad\qquad F[\Sigma] = \int e^{-x^2/4} d\mu_\Sigma .$$

Essentially, the Bernstein-type theorems say that the only varifold shrinkers that are stable in a halfspace (respectively don't intersect in a halfspace) are flat planes. The precise statements are as follows:[5]

Theorem 3.4 (First Bernstein-type theorem [CHHW19, Thm. 3.8]). *Let Σ be a varifold shrinker with multiplicity one in $\mathbb{R}^{n+1}$, such that $\mathcal{H}^{n-1}(\operatorname{sing} \Sigma \cap \mathbb{H}) = 0$. If $\operatorname{reg} \Sigma \cap \mathbb{H}$ is stable for the F-functional, then Σ is a multiplicity one hyperplane.*

Theorem 3.5 (Second Bernstein-type theorem [CHHW19, Thm. 3.11, Cor. 3.12, Cor. 3.14]). *For $a = 1, 2$ let Σ_a be the support of an n-dimensional varifold shrinker in $\mathbb{R}^{n+1}$ with $\mathcal{H}^{n-1}(\operatorname{sing} \Sigma_a) = 0$. Suppose that $\operatorname{reg} \Sigma_1 \cap \mathbb{H}$ and $\operatorname{reg} \Sigma_2 \cap \mathbb{H}$ are nonempty and do not intersect transversely at any point. Then either*

    (i) $\Sigma_1 = \Sigma_2$, or

    (ii) Σ_1 and Σ_2 are flat planes.

We can now prove the main theorem of this section:

Proof of Theorem 3.1. Consider the collection C of mutual tangent cones at the origin (see Definition 3.2), and suppose that $(\Sigma_1, \Sigma_2) \in C$. Observe that

$$(10) \qquad\qquad \operatorname{reg} \Sigma_1 \cap \mathbb{H} \text{ and } \operatorname{reg} \Sigma_2 \cap \mathbb{H} \text{ do not intersect transversely at any point,}$$

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[5]In [CHHW19, Thm. 3.8] a sharper result has been established, but the simplified version here is sufficient for our purpose.

since otherwise $\operatorname{reg} M_1 \cap \mathbb{H}$ and $\operatorname{reg} M_2 \cap \mathbb{H}$ would intersect, contrary to Hypothesis (iii). Thus, by Theorem 3.5 (Second Bernstein-type theorem), either Σ_1 and Σ_2 are distinct planes or $\Sigma_1 = \Sigma_2$. In the first case, we are done, so (in our argument by contradiction) we can assume from now on that

$$\Sigma_1 = \Sigma_2 \quad \text{for all } (\Sigma_1, \Sigma_2) \in C. \tag{11}$$

Furthermore, in the CMC-case, if M_1 and M_2 curve oppositely at 0 then we are in scenario (a), and thus done, so we can assume from now on that the orientations are consistent.

Suppose towards a contradiction that $(\Sigma, \Sigma) \in C$. Since $\partial B(0, \sqrt{2n})$ is a (smooth) shrinker, it follows from Theorem 3.5 that $\operatorname{reg} \Sigma \cap \mathbb{H}$ and $\partial B(0, \sqrt{2n}) \cap \mathbb{H}$ have nonempty intersection, i.e.

$$\operatorname{reg} \Sigma \cap B(0, \sqrt{2n}) \cap \mathbb{H} \neq \emptyset. \tag{12}$$

Here, we also used that $\Sigma \neq \partial \mathbb{H}$ by hypothesis (ii).

Our goal is now to construct a nontrivial Jacobi-field. To this end, we will first locate regions of approximately maximal regularity scale. For any set $S \subset \mathbb{R}^{n+1}$ and any $p \in S$, denote by $R(S, p)$ be the regularity scale of S at p, i.e. the supremum of $r \geq 0$ such that $S \cap B(p, r)$ is a smooth n-dimensional manifold (without boundary in $B(p, r)$) properly embedded in $B(p, r)$ and such that the norm of the second fundamental form at each point of $S \cap B(p, r)$ is $\leq 1/r$. We consider the quantity

$$\rho(S) := \{\sup R(S \cap \mathbb{H}, p) : p \in S \cap \mathbb{H}, |p| \leq \sqrt{2n}\}. \tag{13}$$

Claim 3.6 (lower bound for regularity scale). *For $a = 1, 2$ we have*

$$\eta_a := \liminf_{\lambda \to \infty} \rho(\lambda M_a) > 0. \tag{14}$$

Proof of Claim 3.6. Choose $\lambda_j \to \infty$ so that $\rho(\lambda_j M_1) \to \eta_1$. By passing to a subsequence, we can assume that $\lambda_j M_1$ converges to a limit Σ. By (12),

$$\rho(\Sigma) > 0. \tag{15}$$

By smooth convergence of $\lambda_j M_1$ to Σ at the regular points, which follows from Allard's regularity theorem [All72], we get

$$\lim_{j \to \infty} \rho(\lambda_j M_1) = \rho(\Sigma). \tag{16}$$

Thus $\eta_1 > 0$. Likewise, $\eta_2 > 0$. This proves the claim. $\qquad\square$

Continuing the proof of the theorem, set[6]

$$\eta := \min\{\eta_1, \eta_2\}. \tag{17}$$

Choose $\Lambda < \infty$ so that $\rho(\lambda M_a) > \eta/2$ for all $\lambda \geq \Lambda$.

For $\lambda \geq \Lambda$, consider the quantity

$$\psi(\lambda) := \sup \left\{ \operatorname{dist}(x, \lambda M_2) : x \in \lambda M_1 \cap \mathbb{H}, |x| \leq \sqrt{2n}, \text{ and } R(\lambda M_1 \cap \mathbb{H}, x) \geq \eta/2 \right\}. \tag{18}$$

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[6]Using (11), it is not hard to show that $\eta_1 = \eta_2$, but we do not need that fact.

Note that the supremum will be attained at some (not necessarily unique) point x_λ. By (11) we have

(19)
$$\lim_{\lambda \to \infty} \psi(\lambda) = 0.$$

Choose $\lambda_j \geqslant j$ so that

(20)
$$\psi(\lambda_j) \geqslant (1 - j^{-1}) \sup_{\lambda \geqslant \lambda_j} \psi(\lambda).$$

By passing to a subsequence, we can assume that $\lambda_j M_1$ converges to a limit Σ and that x_{λ_j} converges to a point $x \in \operatorname{reg} \Sigma \cap \mathbb{H}$. Again by (11), the blowup sequence $\lambda_j M_2$ also converges to Σ. Note that the convergence is smooth on compact subsets of $\mathbb{R}^{n+1} \setminus \operatorname{sing} \Sigma$.

Choose open subsets Ω_j of $\mathbb{H} \cap \operatorname{reg} \Sigma$ such that

$$x \in \Omega_1 \subset \Omega_2 \subset \ldots,$$
$$\Omega_j \subset\subset \operatorname{reg} \Sigma \cap \mathbb{H},$$
$$\cup_j \Omega_j = \operatorname{reg} \Sigma \cap \mathbb{H}.$$

Since $\operatorname{reg} \Sigma \cap \mathbb{H}$ is connected (by Theorem 3.5), we can choose the Ω_j to be connected.

Let ν be a unit normal vectorfield on $\operatorname{reg} \Sigma$. By the smooth convergence of $\lambda_j M_a$ on the regular portion of Σ, we can, by passing to a subsequence, assume that for $a = 1, 2$ there are functions

(21)
$$u_j^a : \Omega_j \to \mathbb{R}$$

such that

(22)
$$\{p + u_j^a(p)\nu(p) : p \in \Omega_j\} \subset \lambda_j M_a,$$

and such that u_j^a converges to 0 smoothly on compact subsets of

(23)
$$\operatorname{reg} \Sigma \cap \mathbb{H}.$$

By relabelling, we may assume that $u_j^2 > u_j^1$ on Ω_j. Then, by the Harnack inequality, and since M_1 and M_2 are consistently oriented at 0 in the CMC case, the renormalized sequence

(24)
$$\frac{u_j^2 - u_j^1}{u_j^2(x_{\lambda_j}) - u_j^1(x_{\lambda_j})}$$

converges smoothly (perhaps after passing to a further subsequence) to a positive solution

(25)
$$u : \operatorname{reg} \Sigma \cap \mathbb{H} \to \mathbb{R}_+$$

of the linearization of the minimal surface equation

(26)
$$\Delta u + |A|^2 u = 0.$$

Moreover, by construction,

(27)
$$u(x) = 1,$$

and, thanks to (20), we have

(28) $\quad |u(\lambda p)| \leqslant \lambda \quad$ for all $\lambda \in (0, 1)$ and all $p \in B(0, \sqrt{2n}) \cap \mathbb{H}$ with $R(\Sigma \cap \mathbb{H}, p) > \eta/2$.

Claim 3.7 (stability). $\operatorname{reg} \Sigma \cap \mathbb{H}$ *is a stable critical point of the F-functional.*

Proof of Claim 3.7. Since Σ is a minimal cone, it is a critical point of the area functional as well as a critical point of the F-functional. Regarding the second variation, recall that the Jacobi operator for the area functional is

$$L = \Delta + |A|^2, \tag{29}$$

and the Jacobi operator for the F-functional is

$$\mathcal{L} = \Delta + |A|^2 - \tfrac{1}{2} x^\top \cdot \nabla + \tfrac{1}{2}. \tag{30}$$

The existence of a positive function u in the kernel of L immediately implies stability with respect to the area functional, see e.g. [FCS80], but does not directly yield stability with respect to the F-functional. To establish the latter, we consider the function

$$w(x, t) = e^{t/2} u(e^{-t/2} x), \tag{31}$$

where $x \in \operatorname{reg} \Sigma \cap \mathbb{H}$ and $t \geqslant 0$. Note that this is well defined, since $\operatorname{reg} \Sigma \cap \mathbb{H}$ is a cone. The function w satisfies

$$Lw = 0, \tag{32}$$

and

$$\partial_t w = \tfrac{1}{2} w - \tfrac{1}{2} x^\top \cdot \nabla w, \tag{33}$$

hence in particular

$$\partial_t w = \mathcal{L} w. \tag{34}$$

Moreover, the growth condition (28) implies that

$$|w(x, t)| \leqslant 1 \quad \text{for all } t \geqslant 0 \text{ and all } x \in B(0, \sqrt{2n}) \cap \mathbb{H} \text{ with } R(\Sigma \cap \mathbb{H}, x) > \eta/2. \tag{35}$$

By the parabolic Harnack inequality this implies

$$\sup_{K \times [1,\infty)} w < \infty \quad \text{for } K \subset\subset \operatorname{reg} \Sigma \cap \mathbb{H}. \tag{36}$$

Let $U \subset\subset \operatorname{reg} \Sigma \cap \mathbb{H}$ be a connected open set. Let λ be the first Dirichlet eigenvalue of $\mathcal{L}$ on U, and let f be the corresponding eigenfunction. Then f is nonzero at all points of U. By multiplying by a constant, we can assume that $0 < f \leqslant w(\cdot, 0)$. Thus, by the maximum principle, we get

$$e^{-\lambda t} f(\cdot) \leqslant w(\cdot, t) \tag{37}$$

for all $t \geqslant 0$. Together with (35) this yields that $\lambda \geqslant 0$. Since U was arbitrary, this proves the claim. □

Thus, by Theorem 3.4 (First Bernstein-type theorem), Σ is a flat plane. Together with the tameness assumption and Allard's regularity theorem [All72], this implies that M_1 and M_2 are smooth at 0. Hence, the classical Hopf lemma in the smooth setting gives $\Sigma_1 \neq \Sigma_2$, contradicting our assumption (11). This concludes the proof of the theorem. □

4. MOVING PLANES FOR COMPACT VARIFOLDS WITH SMOOTH BOUNDARY

In this section, we implement our version of the moving plane method in the setting of compact varifolds with smooth boundary. Theorem 1.4 and Theorem 1.9 are proved simultaneously in Theorem 4.1 and Corollary 4.4 below.

Theorem 4.1 (compact varifolds with smooth boundary). *Suppose $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are smooth, closed, strictly convex $(n - 1)$-dimensional surfaces in parallel n-planes Q_1 and Q_2 in $\mathbb{R}^{n+1}$. Suppose that M is a compact connected integral n-varifold that is stationary or CMC in $\mathbb{R}^{n+1} \backslash (\Gamma_1 \cup \Gamma_2)$ and such that the density of M is $1/2$ at each point of Γ. In the stationary case, we assume that M has no triple junctions. In the CMC-case, we assume that M is tame, lies in the slab bounded by $Q_1 \cup Q_2$ and meets Γ transversally.*

If P is a hyperplane of symmetry of the boundary Γ, then P is a plane of symmetry of M. Moreover, the portion of M on each side of P is a smooth graph over a region in P.

Before starting the moving plane argument, let us record the following basic property:

Proposition 4.2 (enclosed domain and mean curvature). *Let D_1, D_2 be the convex domains in Q_1, Q_2 bounded by Γ_1 and Γ_2. Then $M \cup D_1 \cup D_2$ bounds a compact set $K \subseteq \mathbb{R}^{n+1}$. Moreover, there exists some $h \in \mathbb{R}$, such that $\mathbf{H} = h\nu$ for every $x \in \text{reg } M$, where ν denotes the outward unit normal.*

Proof. By Lemma 4.5 below in the stationary case (respectively by assumption in the CMC-case), the varifold M is tame. Hence, by Definition 1.6 (tameness) any tangent cone to M that splits-off an $\mathbb{R}^{n-1}$-factor must be a multiplicity-one hyperplane. By standard stratification (see e.g. [Sim83]) this implies

$$\tag{38} \mathcal{H}^{n-1}(\text{sing } M) = 0.$$

Since sing $M \subset \mathbb{R}^{n+1}$ has codimension bigger than 2, we have the following two properties:

- any smooth curve in $\mathbb{R}^{n+1}$ can be perturbed such that it avoids sing M,
- any smooth 2-disc in $\mathbb{R}^{n+1}$ bounding a closed curve can be perturbed such that it avoids sing M and meets reg M transversally.

Hence, by standard intersection theory (see e.g. [Sam69]) the set $M \cup D_1 \cup D_2$ encloses a compact set $K \subseteq \mathbb{R}^{n+1}$, namely

$$\tag{39} \partial K = M \cup D_1 \cup D_2.$$

Finally, since M is connected by assumption, using again (38) we see that reg M is connected, from which the second assertion follows. $\square$

After these preparations, we can now implement our new moving plane method:

Proof of Theorem 4.1. We will focus on the situation where P is perpendicular to Q_a (the argument when P is parallel to Q_a is similar). We can assume without loss of generality that the planes Q_1 and Q_2 are parallel to the plane $\{x_{n+1} = 0\}$ and that P is the plane $\{x_1 = 0\}$.

For $s \geqslant 0$ we let

$$M_s^- = \{x \in M \; : \; x_1 < s\},$$
$$M_s^+ = \{x \in M \; : \; x_1 > s\},$$

and we let M_s^* be the image of M_s^+ under reflection in the plane $\{x_1 = s\}$. Similarly, we let

$$K_s^- = \{x \in K \; : \; x_1 < s\},$$
$$K_s^+ = \{x \in K \; : \; x_1 > s\},$$

and we let K_s^* be the image of K_s^+ under reflection in the plane $\{x_1 = s\}$.

Let $\mathcal{S}$ be the set of $s > 0$ such that

(i) In $M \cap \{x_1 \geqslant s\}$, every point is a regular point and $\mathbf{e}_1 \cdot \nu > 0$, where ν is the unit normal to M that points out of K.

(ii) Each line parallel to the x_1-axis intersects M_s^+ in at most one point.

(iii) $K_s^* \subseteq K_s^-$.

(iv) M_s^* and M_s^- are disjoint.

Note that $\mathcal{S}$ is open and nonempty, and that if $s \in \mathcal{S}$, then every $s' > s$ is also in $\mathcal{S}$. Thus

(40) $$\mathcal{S} = (t, \infty),$$

where $t := \inf \mathcal{S} \geqslant 0$.

From (i)–(iv), we see that

(i)′ In M_t^+, every point is a regular point and $\mathbf{e}_1 \cdot \nu > 0$.

(ii)′ Each line parallel to the x_1-axis intersects M_t^+ in at most one point.

(iii)′ $K_t^* \subseteq K_t^-$.

Claim 4.3. $t = 0$.

Proof of the claim. If M_t^* touched M_t^- at some point q, then by the Solomon-White maximum principle (Theorem 1.11), the entire connected component of M_t^* containing q would lie in M_t^-. By unique continuation and by connectedness of reg M, it follows that the plane $\{x_1 = t\}$ is a plane of symmetry of M. Therefore it is a plane of symmetry of Γ, and hence $t = 0$ as claimed.

Thus in proving the claim, we may assume that M_t^* and M_t^- are disjoint.

Suppose that the claim is false, i.e., that $t > 0$. By the smooth Hopf lemma, $\mathbf{e}_1 \cdot \nu > 0$ at each regular point of $M \cap \{x_1 = t\}$. Thus, since $t \notin \mathcal{S}$, there must be a singular point p in $M \cap \{x_1 = t\}$. By the Allard's boundary regularity Theorem [All75], p is not in Γ.

Let K' and M' be the images of K and M under reflection in the plane $\{x_1 = t\}$. In the CMC case, the fact that $\mathbf{H} = h\nu$ on reg$M$ (see Proposition 4.2) implies that if (Σ, Σ) is a mutual tangent cone to M and M' at p, then the induced orientations on Σ from M and M' are compatible (they are either the inward pointing or outward pointing, according to the sign of h). Thus, M and M' are **not** oppositely oriented at p. Letting Σ be any tangent cone to M at p, applying the Hopf lemma for varifolds (Theorem 1.10) to M, M', the point p, and the half space $\{x_1 < t\}$ gives that Σ is a plane, both in the CMC and the stationary case. This contradicts the fact that p is a singular point and thus, completes the proof of the claim. $\qquad \square$

Continuing the proof of the theorem, by the claim we have $t = 0$, and thus $K_0^* \subseteq K_0^-$. The same argument shows that the reflected image of $K \cap \{x_1 < 0\}$ lies in $K \cap \{x_1 > 0\}$. Thus K and M are invariant under reflection in $\{x_1 = 0\}$. This together with (i)$'$ and (ii)$'$ completes the proof of Theorem 4.1. $\square$

Corollary 4.4. *Suppose in Theorem 4.1 that Γ_1 and Γ_2 are $(n-1)$-spheres in parallel n-planes that are rotationally invariant about the same axis. Then M is smooth and rotationally invariant about that axis, and thus is a portion of an n-dimensional catenoid or of a Delaunay hypersurface.*

Proof. We can apply Theorem 4.1 for each n-plane P containing the axis. It follows that M is rotationally invariant about the axis, and smooth except possibly along the axis. If M contained a point on the axis, then the tangent cone would be rotationally invariant about the axis, and thus by tameness would be a multiplicity-one plane. By Allard's Regularity Theorem, it would be a regular point. Thus M is smooth everywhere. This proves the corollary. $\square$

Let us observe that the above argument also yields uniqueness of CMC-spherical caps:

Proof of Theorem 1.7. The proof is similar as above, with the only change that now the boundary has only one component instead of two components. $\square$

The following lemma has been used in the above proof:

Lemma 4.5 (tameness)**.** *Every stationary integral n-varifold M satisfying the assumptions of Theorem 4.1 is tame.*

Proof. Let $x \in M \backslash (Q_1 \cup Q_2)$. For $a = 1, 2$ denote by E_a the exterior cone over Γ_a with vertex x:

$$E_a = \{x + \lambda(y - x) : y \in \Gamma_a\}. \tag{41}$$

Applying the extended monotonicity formula from [EWW02] to $M \cup E_1 \cup E_2$, we get

$$\Theta(M, x) < 2, \tag{42}$$

where we used that E_a has density at infinity strictly less than 1, thanks to convexity. This shows that all tangent cones of M have multiplicity 1.

To show tameness, suppose M had a tangent cone C at some point x, such that C itself had a triple-junction tangent cone T (not at the origin, of course). Then there exists $x_i \to x$ and $\lambda_i \to \infty$ such that $M_i := \lambda_i(M - x_i)$ converges to T in the sense of varifolds. Let γ be a circle of radius 1 around the origin in the cross-sectional plane of the triple junction, and observe that γ intersects T transversally at three regular points. Hence, by Allard's regularity theorem, γ intersects M_i transversally at three regular points, when i is large enough. Thus, there exists a curve $\tilde{\gamma}$ intersecting M transversally at three regular points. If follows that there exists a closed curve γ that intersects M three times transversally in a regular point. However, we have $\mathcal{H}^{n-1}(\text{sing } M) = 0$ since M does not have triple-junction singularities by assumption, so the number of intersections must be even; a contradiction. This proves the lemma. $\square$

To conclude this section, let us explain how to prove Corollary 1.5:

Proof of Corollary 1.5. To see that (i$'$) and (ii$'$) imply (i), (ii), and (iii), note that (i$'$) implies stationarity of M away from Γ. By the convex hull property, any tangent cone to M at any point of Γ lies in a wedge. By Corollary 32 of [Whi19], such a tangent cone is a union of halfplanes. The mod 2 hypothesis (ii$'$) implies that the number k of halfplanes (counting multiplicity) is odd. The inequality (i$'$) then implies that $k = 1$, and thus that the density is $1/2$. (See Theorem 34 of [Whi19] for details.) Finally, (ii$'$) implies that there are no triple junctions. (If there were a triple junction, we could find a small circle that does not link Γ and that intersects M transversely in exactly three points, each of multiplicity one.) □

5. MOVING PLANES FOR COMPLETE VARIFOLDS

In this section, we implement our version of the moving plane method in the setting of complete varifolds and prove Theorem 1.3, which we restate here for convenience of the reader:

Theorem 5.1 (Uniqueness of the catenoid). *Suppose M is stationary complete integral n-varifold in $\mathbb{R}^{n+1}$ such that*

  (i) some tangent cone at infinity is a multiplicity-2 plane,
  (ii) M has at least two ends,
  (iii) M has no triple junctions.

Then M is a smooth hypersurface of revolution, and thus either a pair of parallel planes or an n-dimensional catenoid.

Proof. First observe that assumptions (i) and (ii) together with monotonicity imply that

$$\Theta(M, x) < 2 \tag{43}$$

for every $x \in M$. Thus, the same argument as in the last paragraph of the proof of Lemma 4.5 shows that M is tame. If $n = 2$, then M is smooth and rotational symmetry follows from the classical result of Schoen [Sch83], so we can assume from now on that $n \geqslant 3$. We can also assume that M is connected, since otherwise by monotonicity it must be the union of two parallel planes.

Next, using also Allard's regularity theorem we can find some $R < \infty$ such that $M \backslash B(0, R)$ is regular and consists of graphs of two smooth functions $u_{\pm} : \mathbb{R}^n \backslash B^n(0, R) \to \mathbb{R}$, satisfying

$$\lim_{|x| \to \infty} |\nabla u_{\pm}| = 0. \tag{44}$$

Hence, the graphical minimal surface equation outside a large ball is just a perturbation of Laplace's equation on $\mathbb{R}^n$, which has Green's function $c_n |x|^{2-n}$, and it easily follows that

$$u_{\pm} = a_{\pm} + O(|x|^{2-n}). \tag{45}$$

Using this, the maximum principle for varifolds (Theorem 1.11) implies that

$$M \text{ must be contained in the slab } \{a_- < x_{n+1} < a_+\}, \tag{46}$$

in particular $a_- < a_+$. After translating in x_{n+1}-direction we can assume that $a_- = -a_+$. Moreover, as in Proposition 4.2, we see that M encloses a domain K with one end.

Claim 5.2 (reflection symmetry and smoothness away from plane). *Reflection across $\{x_{n+1} = 0\}$ is a symmetry of M. Moreover, $M \cap \{x_{n+1} \neq 0\}$ is smooth.*

Proof of Claim 5.2. For $s \in (a_-, a_+)$ we let

$$M_s^- = \{x \in M \ : \ x_{n+1} < s\},$$
$$M_s^+ = \{x \in M \ : \ x_{n+1} > s\},$$

and we let M_s^* be the image of M_s^+ under reflection in the plane $\{x_{n+1} = s\}$. Similarly, we let

$$K_s^- = \{x \in K \ : \ x_{n+1} < s\},$$
$$K_s^+ = \{x \in K \ : \ x_{n=1} > s\},$$

and we let K_s^* be the image of K_s^+ under reflection in the plane $\{x_{n+1} = s\}$.

Let $\mathcal{S}$ be the set of $s > 0$ such that

 (i) In $M \cap \{x_{n+1} \geqslant s\}$, every point is a regular point and $\mathbf{e}_{n+1} \cdot \nu > 0$, where ν is the unit normal to M that points out of K.
 (ii) Each line parallel to the x_{n+1}-axis intersects M_s^+ in at most one point.
 (iii) $K_s^* \subset K_s^-$.
 (iv) M_s^* and M_s^- are disjoint.

Note that if $s \in \mathcal{S}$, then every $s' \in [s, a_+)$ is also in $\mathcal{S}$. By (45) and (46), if t is sufficiently close to a_+ then $t \in \mathcal{S}$ and all points in $M \cap \{x_{n+1} \geqslant t\}$ are smooth. This gets the moving plane method started. Moreover, using again (45) and (46) we see that $\mathcal{S}$ is open. Hence,

(47) $$\mathcal{S} = (t, \infty),$$

where $t := \inf \mathcal{S} \geqslant 0$. Suppose towards a contradiction that $t > 0$.

Using the Solomon-White maximum principle (Theorem 1.11) and (45) we see that M_t^* and M_t^- are disjoint. By the smooth Hopf lemma, $\mathbf{e}_{n+1} \cdot \nu > 0$ at each regular point of $M \cap \{x_{n+1} = t\}$. Thus, since $t \notin \mathcal{S}$, there must be a singular point p in $M \cap \{x_{n+1} = t\}$. Let M' be the images of M under reflection across the plane $\{x_{n+1} = t\}$. Letting Σ be any tangent cone to M at p, applying the Hopf lemma for varifolds (Theorem 1.10) to M, M', the point p, and the half space $\{x_{n+1} < t\}$ gives that Σ is a plane. This contradicts the fact that p is a singular point and thus shows that $t = 0$. This implies the assertion. □

Claim 5.3 (rotational symmetry and smoothness away from the axis). *M is rotationally symmetric around the x_{n+1}-axis and smooth away from the x_{n+1}-axis.*

Proof of Claim 5.3. By rotation of coordinates it suffices to consider the moving plane $\{x_1 = t\}$. The argument is similar as above, with the only difference that getting the moving plane method started requires a somewhat more careful expansion at infinity. To this end, recall that by [Sch83, Prop. 3] after a suitable

shift in the $(x_1, \ldots, x_n)$-plane the function u_+ (and by the established reflection symmetry, also $-u_-$) can be expanded as

$$u_+(x) = a - b|x|^{2-n} + O(|x|^{-n}), \tag{48}$$

where $a > 0$, with similar expansions for the derivates. By the maximum principle for varifolds (Theorem 1.11) we have $b \geqslant 0$. Integrating the equation $\Delta_M x_{n+1} = 0$ over a large annulus, and applying the divergence theorem, yields $b > 0$. Using this, we see that for $\varepsilon > 0$ small enough we have that $M \cap \{x_{n+1} = a - \varepsilon\}$ is C^2-close to a round sphere centered at $(0, \ldots, 0, a - \varepsilon)$ with radius

$$R_\varepsilon = \left(\frac{b}{\varepsilon}\right)^{\frac{1}{n-2}}, \tag{49}$$

up to error terms of order $O(R_\varepsilon^{-1})$, and similarly for $M \cap \{x_{n+1} = -a + \varepsilon\}$. This shows that there is no contact at infinity and gets the moving plane method started. Pushing the moving plane as in the proof of the previous claim, this yields the assertion. $\qquad\square$

By the above two claims, M is reflection symmetric and rotationally symmetric and smooth, except possibly at the origin. If the origin where contained in M, then by rotational symmetry and tameness the tangent cone there would be a multiplicity-one plane. Thus, M is smooth, and hence an n-dimensional catenoid. This finishes the proof of the theorem. $\qquad\square$

6. THE MOVING PLANE METHOD ON MANIFOLDS WITH SYMMETRY

The goal of this section is to prove Theorem 1.8. In fact, our entire discussion so far readily generalizes to Riemannian manifolds which are symmetric with respect a moving plane.

Theorem 6.1. *Consider a smooth Riemannian metric on $N = \mathbb{R} \times N_0$ such that*

(i) *For every c, the metric is invariant under the reflection F_c, defined by $(x_1, p) \mapsto (2c - x_1, p)$, and*

(ii) *The vectorfield ∂_{x_1} is orthogonal to the foliation $\{x_1 = c\}$.*

Suppose M is a compact connected tame CMC-varifold in N without boundary. Then M is invariant under reflection across some $N_c := \{x_1 = c\}$. Furthermore, the portion of M on either side of that N_c is a smooth graph over an open subset of N_c.

Proof. The proof of the Hopf Lemma for varifolds (Theorem 1.10) and the strong maximum principle for varifolds (Theorem 1.11) are still valid in the context of any Riemannian manifold. The symmetry assumptions $(i), (ii)$ allows one to argue precisely as in the proof of Theorem 4.1, to show that $\mathcal{S} = (c, \infty)$ for some c such that M is invariant under reflection across N_c. $\qquad\square$

Note that if $||\partial_{x_1}|| = 1$ in Theorem 6.1, then assumptions $(i), (ii)$ imply that N carries a product metric. In general, the metric on N need not split an $\mathbb{R}$-factor, as we shall now exploit:

Lemma 6.2. *There exists a model to the hyperbolic space of the form $N = \mathbb{R} \times N_0$, satisfying (i) and (ii).*

Proof. Consider the upper half space model on $\mathbb{H}^{n+1}$, with the metric $g = \frac{dx_1^2 + \dots + dx_{n+1}^2}{x_1^2}$, and let $N = \mathbb{R} \times \mathbb{S}_+^n$, where $\mathbb{S}_+^n$ is the upper hemisphere of n-th sphere. Considering the diffeomorpishm $f : N \to \mathbb{H}^{n+1}$ defined by

$$f(t, \omega) = e^t \omega, \tag{50}$$

one easily sees that the metric $f^* g$ is t-independent, and is such that ∂_t is orthogonal to the hypersurfaces $\{t = c\}$. □

Proof of Theorem 1.8. Let us first consider the case $M \subset \mathbb{H}^{n+1}$. Since M is compact, there exists a unique ball of minimal radius $B(p, R) \subset \mathbb{H}^{n+1}$ containing M. For each n-dimensional hyperplane in $P \subseteq T_p\mathbb{H}^{n+1}$, we have that $\tilde{P} := \{\exp_p(v), \ v \in P\}$ is a totally geodesic hypersurface in $\mathbb{H}^{n+1}$. Let Ψ be an isometry of $\mathbb{H}^{n+1}$ sending p to some point $q \in N_0$, such that $d\Psi_p(P) = T_q N_0$. Since N_0 is also totally geodesic, we see that Ψ sends $\tilde{P}$ to N_0. Now, Theorem 6.1, which can be applied thanks to Lemma 6.2, implies that there exists some $c \in \mathbb{R}$ such that the $F_c(\Psi(M)) = \Psi(M)$, so $M = \Psi^{-1}(F_c(\Psi(M)))$. As the isometry $\Phi := \Psi^{-1} \circ F_c \circ \Psi$ fixes M, it also has to fix the ball $B(p, R)$, and so it fixes p. Thus $c = 0$. Note further that $d\Phi_p$ acts on $T_p\mathbb{H}^{n+1}$ by a reflection about $\tilde{P}$. As reflections generate $\mathrm{SO}(n + 1)$, and as isometries of $\mathbb{H}^{n+1}$ fixing p are uniquely determined by their differential at p, we see that M is smooth, and is fixed by all of the isometries of $\mathbb{H}^{n+1}$ that fix p. Since M is connected, we conclude that M is a a geodesic sphere.

For $M \subseteq \mathbb{H}^n \times \mathbb{R}$, the same argument shows that there exists some point $(p, x) \in \mathbb{H}^n \times \mathbb{R}$ such that M is invariant under all isometries of the form (Ψ, Id), where Ψ is an isometry of $\mathbb{H}^n$ fixing p. Thus, M is rotationally symmetric around $\{p\} \times \mathbb{R}$ and is smooth away from the axis. Arguing as in the proof of Corollary 4.4 we also get smoothness along the axis of symmetry. □

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