LECTURES ON MEAN CURVATURE FLOW OF SURFACES

ROBERT HASLHOFER

Abstract. Mean curvature flow is the most natural evolution equation in extrinsic geometry, and shares many features with Hamilton's Ricci flow from intrinsic geometry. In this lecture series, I will provide an introduction to the mean curvature flow of surfaces, with a focus on the analysis of singularities. We will see that the surfaces evolve uniquely through neck singularities and nonuniquely through conical singularities. Studying these questions, we will also learn many general concepts and methods, such as monotonicity formulas, epsilon-regularity, weak solutions, and blowup analysis that are of great importance in the analysis of a wide range of partial differential equations. These lecture notes are from the 2021 summer school in PDE at UT Austin, and also contain a detailed discussion of open problems and conjectures.

Contents

Date: May 21, 2021.

I thank Francesco Maggi and Phil Isett for organizing the 2021 summer school at UT Austin, and all the participants for their questions and feedback. Some of the results described in these lectures are from joint work with Kyeongsu Choi, Wenkui Du, Or Hershkovits, Bruce Kleiner and Brian White. My research has been supported by an NSERC Discovery Grant and a Sloan Research Fellowship.

1. Overview and basic properties

In this first lecture, I will give a quick informal introduction to the mean curvature flow of surfaces.[1]

A smooth family of embedded surfaces $\{M_t \subset \mathbb{R}^3\}_{t\in I}$ moves by mean curvature flow if

$$\partial_t x = \vec{H}(x) \tag{1.1}$$

for $x \in M_t$ and $t \in I$. Here, $I \subset \mathbb{R}$ is an interval, $\partial_t x$ is the normal velocity at x, and $\vec{H}(x)$ is the mean curvature vector at x.

If we write $\vec{H} = H\vec{\nu}$, where $\vec{\nu}$ is the inwards unit normal, then H is given by the sum of the principal curvatures, $H = \kappa_1 + \kappa_2$. Recall that the principal curvatures are the eigenvalues of the second fundamental form. More concretely, given any point p on a surface M, if we express the surface locally as a graph of a function u over the tangent space T_pM, then $\kappa_1(p)$ and $\kappa_2(p)$ are simply the eigenvalues of $\mathrm{Hess}(u)(p)$.

Example 1.2 (Shrinking sphere and cylinder)**.** If $M_t = S^2(r(t))$ is a round sphere, then equation (1.1) reduces to an ODE for the radius, namely $\dot{r} = -2/r$. The solution with $r(0) = R$ is $r(t) = \sqrt{R^2 - 4t}$, where $t \in (-\infty, R^2/4)$. Similarly, we have the round shrinking cylinder $M_t = \mathbb{R} \times S^1(r(t))$ with $r(t) = \sqrt{R^2 - 2t}$, where $t \in (-\infty, R^2/2)$.

Exercise 1.3 (Graphical evolution)**.** *Show that if* $M_t = \mathrm{graph}(u(\cdot, t))$ *is the graph of a time-dependent function* $u(\cdot, t) : \mathbb{R}^2 \to \mathbb{R}$*, then*

$$\partial_t u = \sqrt{1 + |Du|^2}\,\mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right). \tag{1.4}$$

Instead of viewing the mean curvature flow as an evolution equation for the hypersurfaces M_t, we can also view it as an evolution equation for a smooth family of embeddings $X : M^2 \times I \to \mathbb{R}^3$ with $M_t = X(M, t)$. Setting $x = X(p, t)$, equation (1.1) then takes the form

$$\partial_t X(p, t) = \Delta_{M_t} X(p, t). \tag{1.5}$$

The fundamental idea of geometric flows is to deform a given geometric object into a nicer one, by evolving it by a heat-type equation. This indeed works very well, as illustrated by the following theorem.

[1]For concreteness we focus on 2-dimensional surfaces in $\mathbb{R}^3$, but of course many things could be generalized to higher dimensions and other ambient spaces.

Theorem 1.6 (Huisken's convergence theorem)**.** *Let $M_0 \subset \mathbb{R}^3$ be a closed embedded surface. If M_0 is convex, then the mean curvature flow $\{M_t\}_{t\in[0,T)}$ starting at M_0 converges to a round point.*

The convex case ($\kappa_1 \geq 0$ and $\kappa_2 \geq 0$) is of course very special. In more general situations, we encounter the formation of singularities.

Example 1.7 (Neckpinch singularity)**.** If M_0 has the topology of a sphere but the geometry of a dumbbell, then the neck pinches off. As blowup limit we get a selfsimilarly shrinking round cylinder. There is also a degenerate variant of this example, where as blowup limit along suitable tip points one gets the selfsimilarly translating bowl soliton.

In the study of mean curvature flow (and indeed of most nonlinear PDEs) it is of crucial importance to understand singularities:

- How do singularities look like?
- Can we continue the flow through singularities?
- What is the size and the structure of the singular set?
- Is the evolution through singularities unique or nonunique?

The analysis of singularities will be our main focus for the following lectures. We conclude this first lecture, by summarizing a few basic properties of the mean curvature flow.

First, by standard parabolic theory, given any compact initial hypersurface $M_0 \subset \mathbb{R}^3$ (say smooth and embedded), there exists a unique smooth solution $\{M_t\}_{t\in[0,T)}$ of (1.1) starting at M_0, and defined on a maximal time interval $[0,T)$. The maximal time T is characterized by the property that the curvature blows up, i.e. $\lim_{t\to T} \max_{M_t}|A| = \infty$.

Second, like for any second order parabolic equation, time scales like distance squared. For example, if $u(x,t)$ solves the heat equation $\partial_t u = \Delta u$, then given any $\lambda > 0$ the parabolically rescaled function $u^\lambda(x,t) = u(\lambda x, \lambda^2 t)$ again solves $\partial_t u^\lambda = \Delta u^\lambda$. The following exercise shows that the same rescaling indeed works for mean curvature flow:

Exercise 1.8 (Parabolic rescaling)**.** *Let $M_t \subset \mathbb{R}^3$ be a mean curvature flow of surfaces, and let $\lambda > 0$. Let $M^\lambda_{t'}$ be the family of surfaces obtained by the parabolic rescaling $x' = \lambda x$, $t' = \lambda^2 t$, i.e. let $M^\lambda_{t'} = \lambda M_{\lambda^{-2}t'}$. Show that $M^\lambda_{t'}$ again solves* (1.1).

In particular, due to the scaling, all estimates naturally take place in parabolic balls $P(x_0, t_0, r) = B(x_0, r) \times (t_0 - r^2, t_0]$.

Third, by the maximum principle (aka avoidance principle) mean curvature flows do not bump into each other. More precisely, if M_t and

N_t are two mean curvature flows (say at least one of them compact), then $\mathrm{dist}(M_t, N_t)$ is nondecreasing in time. In particular, if M_{t_0} and N_{t_0} are disjoint, then so are M_t and N_t for all $t \geq t_0$. Similarly, the flow does not bump into itself, i.e. embeddedness is also preserved.

Forth, the evolution equation (1.1) implies evolution equations for the induced metric g_{ij}, the area element $d\mu$, the normal vector $\vec{\nu}$, the mean curvature H, and the second fundamental form A:

Proposition 1.9 (Evolution equations for geometric quantities)**.** *If $M_t \subset \mathbb{R}^3$ evolves by mean curvature flow, then*

$$\begin{array}{lll} \partial_t g_{ij} = -2HA_{ij} & \partial_t d\mu = -H^2 d\mu & \partial_t \vec{\nu} = -\nabla H \\ \partial_t H = \Delta H + |A|^2 H & \partial_t A^i_j = \Delta A^i_j + |A|^2 A^i_j. & \end{array} \tag{1.10}$$

For example, the evolution of $g_{ij} = \partial_i X \cdot \partial_j X$ is computed via

$$\partial_t g_{ij} = 2\partial_i (H\vec{\nu}) \cdot \partial_j X = 2H \partial_i \vec{\nu} \cdot \partial_j X = -2HA_{ij}. \tag{1.11}$$

Exercise 1.12 (Evolution of the area element)**.** *Show that if $G = G(t)$ is a smooth family of invertible matrices, then $\frac{d}{dt} \ln \det G = tr_G \frac{d}{dt} G$. Use this to derive the evolution equation for $d\mu = \sqrt{\det g_{ij}}\, d^2x$.*

In particular, if M_0 is compact the total area decreases according to

$$\frac{d}{dt}\mathrm{Area}(M_t) = -\int_{M_t} H^2 d\mu. \tag{1.13}$$

One can think of this as a variational characterization of the mean curvature flow as the gradient flow of the area functional.

Finally, using Proposition 1.9 and the maximum principle we obtain:

Corollary 1.14 (mean-convexity and convexity)**.** *Let $M_t \subset \mathbb{R}^3$ be a mean curvature flow of closed surfaces. If $H \geq 0$ at $t = 0$, then $H \geq 0$ for all $t > 0$. Similarly, convexity is also preserved.*

We note that convexity, i.e. $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$, is a much stronger assumption than mean-convexity, i.e. $H = \kappa_1 + \kappa_2 \geq 0$.[2] Mean-convexity is on the one hand is flexible enough to allow for interesting singularities, e.g. the neck-pinch and degenerate neck-pinch, and flexible enough for interesting applications[3], but on the other hand, as we will see, rigid enough to obtain a detailed description of singularities.

[2]Compare this with the study of convex functions, i.e. functions satisfying $\mathrm{Hess}\, u \geq 0$, versus subharmonic functions, i.e. functions satisfying $\Delta u \geq 0$.

[3]The presumably most famous application for inverse mean curvature flow is the proof of the Penrose inequality by Huisken-Ilmanen [HI01]. For some other applications of mean-convex flows see e.g. [Sch08, HS09, BHH, HK19, LM20].

Related PDEs. The mean curvature flow is closely related to other geometric PDEs, including in particular the harmonic map flow $\partial_t u = \Delta_{M,N} u$, Hamilton's Ricci flow $\partial_t g_{ij} = -2\mathrm{Rc}_{ij}$ and the Yang-Mills flow $\partial_t A = -D_A^* F_A$. It is very fruitful to study these PDEs in parallel, as new insights on one of them often leads to progress on the others.

References. Mean curvature flow first appeared as a model for evolving interfaces in material science [Mul56]. Its mathematical study was pioneered by Brakke [Bra78] and Huisken [Hui84]. Nice textbooks on mean curvature flow include the ones by Ecker [Eck04] and Mantegazza [Man11]. I also recommend the notes from the lectures by White [Whi15] and Schulze [Sch17]. Finally, let me point you to my own lecture notes [Has14, Has16], as well as the video recordings from my topics course at the Fields Institute [Has17].

2. Monotonicity formula and epsilon-regularity

In this second lecture, we discuss Huisken's monotonicity formula and the epsilon-regularity theorem for the mean curvature flow.

Recall that by equation (1.13) the total area is monotone under mean curvature flow. However, since $\mathrm{Area}(\lambda M) = \lambda^2 \mathrm{Area}(M)$, this is not that useful when considering blowup sequences with $\lambda \to \infty$. A great advance was made by Huisken, who discovered a scale invariant monotone quantity. To describe this, let $\mathcal{M} = \{M_t \subset \mathbb{R}^3\}$ be a smooth mean curvature flow of surfaces, say with at most polynomial volume growth, let $X_0 = (x_0, t_0)$ be a point in space-time, and let

$$\rho_{X_0}(x,t) = \frac{1}{4\pi(t_0 - t)} e^{-\frac{|x-x_0|^2}{4(t_0-t)}} \qquad (t < t_0), \tag{2.1}$$

be the 2-dimensional backwards heat kernel centered at X_0.

Theorem 2.2 (Huisken's monotonicity formula)**.**

$$\frac{d}{dt}\int_{M_t} \rho_{X_0} d\mu = -\int_{M_t} \left| \vec{H} - \frac{(x-x_0)^\perp}{2(t-t_0)} \right|^2 \rho_{X_0} d\mu \qquad (t < t_0). \tag{2.3}$$

Huisken's monotonicity formula (2.3) can be thought of as weighted version of (1.13). A key property is its invariance under rescaling.

Exercise 2.4 (Scaling invariance)**.** *Let* $x' = \lambda(x - x_0)$, $t' = \lambda^2(t - t_0)$, *and consider the rescaled flow* $M^\lambda_{t'} = \lambda(M_{t_0+\lambda^{-2}t'} - x_0)$. *Prove that*

$$\int_{M_t} \rho_{X_0}(x,t)\, d\mu_t(x) = \int_{M^\lambda_{t'}} \rho_0(x',t')\, d\mu_{t'}(x') \qquad (t'<0). \tag{2.5}$$

Another key property is that the equality case of (2.3) exactly characterizes the selfsimilarly shrinking solutions:

Exercise 2.6 (Shrinkers)**.** *Let* $\{M_t \subset \mathbb{R}^3\}_{t\in(-\infty,0)}$ *be an ancient solution of the mean curvature flow. Prove that* $\vec{H} - \frac{x^\perp}{2t} = 0$ *for all* $t<0$ *if and only if* $M_t = \sqrt{-t}M_{-1}$ *for all* $t<0$.

Proof of Theorem 2.2. We may assume without loss of generality that $X_0 = (0,0)$. The proof of Huisken's monotonicity formula essentially amounts to deriving belows pointwise identity (2.9) for $\rho = \rho_0$.

Since the tangential gradient of ρ is given by $\nabla^{M_t}\rho = D\rho - (D\rho\cdot\vec{\nu})\vec{\nu}$, the intrinsic Laplacian of ρ can be expressed as

$$\Delta_{M_t}\rho = \mathrm{div}_{M_t}\nabla^{M_t}\rho = \mathrm{div}_{M_t}D\rho + \vec{H}\cdot D\rho. \tag{2.7}$$

Observing also that $\frac{d}{dt}\rho = \partial_t\rho + \vec{H}\cdot D\rho$, we compute

$$\begin{aligned}(\tfrac{d}{dt} + \Delta_{M_t})\rho &= \partial_t\rho + \mathrm{div}_{M_t}D\rho + 2\vec{H}\cdot D\rho\\ &= \partial_t\rho + \mathrm{div}_{M_t}D\rho + \frac{|\nabla^\perp\rho|^2}{\rho} - |\vec{H} - \frac{\nabla^\perp\rho}{\rho}|^2\rho + H^2\rho. \end{aligned}\tag{2.8}$$

We can now easily check that $\partial_t\rho + \mathrm{div}_{M_t}D\rho + \frac{|\nabla^\perp\rho|^2}{\rho} = 0$. Thus

$$(\tfrac{d}{dt} + \Delta_{M_t} - H^2)\rho = -|\vec{H} - \frac{x^\perp}{2t}|^2\rho. \tag{2.9}$$

Using also the evolution equation $\frac{d}{dt}d\mu = -H^2 d\mu$, we conclude that

$$\frac{d}{dt}\int_{M_t}\rho\, d\mu = -\int_{M_t}\left|\vec{H} - \frac{x^\perp}{2t}\right|^2\rho\, d\mu \qquad (t<0). \tag{2.10}$$

This proves the theorem. □

More generally, if M_t is only defined locally, say in $B(x_0,\sqrt{8}\rho)\times(t_0-\rho^2,t_0)$, then we can localize using the cutoff function

$$\chi^\rho_{X_0}(x,t) = \left(1 - \frac{|x-x_0|^2 + 4(t-t_0)}{\rho^2}\right)^3_+ . \tag{2.11}$$

Observing that $(\frac{d}{dt} - \Delta_{M_t})\chi^\rho_{X_0} \leq 0$ we still get the monotonicity inequality

$$\frac{d}{dt}\int_{M_t} \rho_{X_0}\chi^\rho_{X_0} d\mu \leq -\int_{M_t} \left| \vec{H} - \frac{(x-x_0)^\perp}{2(t-t_0)} \right|^2 \rho_{X_0}\chi^\rho_{X_0} d\mu. \tag{2.12}$$

The monotone quantity appearing on the left hand side,

$$\Theta^\rho(\mathcal{M}, X_0, r) := \int_{M_{t_0-r^2}} \rho_{X_0}\chi^\rho_{X_0} d\mu, \tag{2.13}$$

is called the Gaussian density ratio. Note that $\Theta^\infty(\mathcal{M}, X_0, r) \equiv 1$ for all $r > 0$ if and only if $\mathcal{M}$ is a multiplicity one plane containing X_0.

We will now discuss the epsilon-regularity theorem for the mean curvature flow, which gives definite curvature bounds in a neighborhood of definite size, provided the Gaussian density ratio is close to one.

Theorem 2.14 (epsilon-regularity)**.** *There exist universal constants $\varepsilon > 0$ and $C < \infty$ with the following significance. If $\mathcal{M}$ is a smooth mean curvature flow in a parabolic ball $P(X_0, 8\rho)$ with*

$$\sup_{X \in P(X_0, r)} \Theta^\rho(\mathcal{M}, X, r) < 1 + \varepsilon \tag{2.15}$$

for some $r \in (0, \rho)$, then

$$\sup_{P(X_0, r/2)} |A| \leq Cr^{-1}. \tag{2.16}$$

Note also that if $\Theta < 1 + \frac{\varepsilon}{2}$ holds at some point and some scale, then $\Theta < 1 + \varepsilon$ holds at all nearby points and somewhat smaller scales.

Proof. Suppose the assertion fails. Then there exist a sequence of smooth flows $\mathcal{M}^j$ in $P(0, 8\rho_j)$, for some $\rho_j > 1$, with

$$\sup_{X \in P(0,1)} \Theta^{\rho_j}(\mathcal{M}^j, X, 1) < 1 + j^{-1}, \tag{2.17}$$

but such that there are points $X_j \in P(0, 1/2)$ with $|A|(X_j) > j$.

Using the so-called point selection technique, we can find space-time points $Y_j \in P(0, 3/4)$ with $Q_j = |A|(Y_j) > j$ such that

$$\sup_{P(Y_j, j/10Q_j)} |A| \leq 2Q_j. \tag{2.18}$$

Let us explain how the point selection works: Fix j. If $Y_j^0 = X_j$ already satisfies (2.18) with $Q_j^0 = |A|(Y_j^0)$, we are done. Otherwise, there is a point $Y_j^1 \in P(Y_j^0, j/10Q_j^0)$ with $Q_j^1 = |A|(Y_j^1) > 2Q_j^0$. If Y_j^1 satisfies (2.18), we are done. Otherwise, there is a point $Y_j^2 \in P(Y_j^1, j/10Q_j^1)$

with $Q_j^2 = |A|(Y_j^2) > 2Q_j^1$, etc. Note that $\frac{1}{2} + \frac{j}{10Q_j^0}(1 + \frac{1}{2} + \frac{1}{4} + \ldots) < \frac{3}{4}$. By smoothness, the iteration terminates after a finite number of steps, and the last point of the iteration lies in $P(0, 3/4)$ and satisfies (2.18).

Continuing the proof of the theorem, let $\hat{\mathcal{M}}^j$ be the flows obtained by shifting Y_j to the origin and parabolically rescaling by $Q_j = |A|(Y_j) \to \infty$. Since the rescaled flow satisfies $|A|(0) = 1$ and $\sup_{P(0,j/10)}|A| \leq 2$, we can pass smoothly to a nonflat global limit. On the other hand, by the rigidity case of (2.12), and since

$$\Theta^{\hat{\rho}_j}(\hat{\mathcal{M}}^j, 0, Q_j) < 1 + j^{-1}, \tag{2.19}$$

where $\hat{\rho}_j = Q_j \rho_j \to \infty$, the limit is a flat plane; a contradiction. □

Related PDEs. Monotonicity formulas and epsilon-regularity theorems are a key tool in the study of many PDEs. Historically, this goes back at least to the classical monotonicity formula for harmonic functions. Let me mention a few further instances. The monotonicity formula and epsilon-regularity theorem for the harmonic map flow are due to Struwe [Str88]. The elliptic cousin of Huisken's monotonicity formula is the monotonicity formula for minimal surfaces, which states that if M satisfies $H = 0$ then the function $r \mapsto \frac{\mathrm{Area}(M \cap B_r(p_0))}{\pi r^2}$ is monotone. The epsilon-regularity theorem for minimal surfaces was proved by Allard [All72]. Similarly, for manifolds with nonnegative Ricci-curvature the volume ratios are monotone (but in the opposite direction) by a result of Bishop-Gromov [Gro99]. A monotonicity formula for the Ricci flow was discovered by Perelman [Per02]. The epsilon-regularity theorem for Einstein metrics was proved by Anderson [And90] and the one for the Ricci flow by Hein-Naber [HN14].

References. The monotonicity formula for the mean curvature flow was discovered by Huisken [Hui90]. The local version from the above remark can be found in Ecker's book [Eck04]. A generalization to weak solutions can be found in Ilmanen's notes [Ilm95]. The epsilon-regularity theorem for mean curvature flow was discovered by Brakke [Bra78]. The presented much simpler proof in the setting of smooth flows and limits thereof is due to White [Whi05]. A careful proof for general weak solutions was given by Kasai-Tonegawa [KT14].

3. Noncollapsing, curvature and convexity estimate

In this third lecture, we discuss the noncollapsing result of Andrews, as well as the local curvature estimate and the convexity estimate.

The following quantitative notion of embeddedness plays a key role in the theory of mean-convex mean curvature flow:

Definition 3.1 (noncollapsing)**.** A closed embedded mean-convex surface $M \subset \mathbb{R}^3$ is called *α-noncollapsed*, if each point $p \in M$ admits interior and exterior balls tangent at p of radius $\alpha/H(p)$.

By compactness, every closed embedded mean-convex initial surface is α-noncollapsed for some $\alpha > 0$. This is preserved under the flow:

Theorem 3.2 (Andrews' noncollapsing theorem)**.** *If the initial surface $M_0 \subset \mathbb{R}^3$ is α-noncollapsed, then so is M_t for all $t \in [0, T)$.*

Proof (sketch). For $x \in M$, the interior ball of radius $r(x) = \alpha/H(x)$ has the center point $c(x) = x + r(x)\nu(x)$. The condition that this is indeed an interior ball is equivalent to the inequality

$$\|y - c(x)\|^2 \geq r(x)^2 \qquad \text{for all } y \in M. \tag{3.3}$$

Observing $\|y - c(x)\|^2 = \|y - x\|^2 - 2r(x)\langle y - x, \nu(x)\rangle + r(x)^2$ and inserting $r(x) = \alpha/H(x)$, the inequality (3.3) can be rewritten as

$$\frac{2\langle y - x, \nu(x)\rangle}{\|y - x\|^2} \leq \frac{H(x)}{\alpha} \qquad \text{for all } y \in M. \tag{3.4}$$

Now, given a mean-convex mean curvature flow $M_t = X(M, t)$ of closed embedded surfaces, we consider the quantity

$$Z^*(x, t) = \sup_{y \neq x} \frac{2\langle X(y, t) - X(x, t), \nu(x, t)\rangle}{\|X(y, t) - X(x, t)\|^2}. \tag{3.5}$$

A rather lengthy computation, which we skip, yields that

$$\partial_t Z^* \leq \Delta Z^* + |A|^2 Z^* \tag{3.6}$$

in the viscosity sense. Together with the evolution equation for the mean curvature, $\partial_t H = \Delta H + |A|^2 H$, this implies

$$\partial_t \frac{Z^*}{H} \leq \Delta \frac{Z^*}{H} + 2\langle \nabla \log H, \nabla \frac{Z^*}{H}\rangle\,. \tag{3.7}$$

Hence, by the maximum principle, if the inequality $Z^*/H \leq 1/\alpha$ holds for $t = 0$, then it also holds for all $t > 0$. This proves interior noncollapsing. Finally, a similar argument shows that the inequality

$$Z_*(x, t) = \inf_{y \neq x} \frac{2\langle X(y, t) - X(x, t), \nu(x, t)\rangle}{\|X(y, t) - X(x, t)\|^2} \geq -\frac{H(x, t)}{\alpha} \tag{3.8}$$

is also preserved, which yields exterior noncollapsing. □

The following estimate gives curvature control on a parabolic ball of definite size starting from a mean curvature bound at a single point:

Theorem 3.9 (local curvature estimate)**.** *For all $\alpha > 0$ there exist $\rho = \rho(\alpha) > 0$ and $C = C(\alpha) < \infty$ with the following significance. If $\mathcal{M}$ is an α-noncollapsed flow defined in a parabolic ball $P(p,t,r)$ centered at a point $p \in M_t$ with $H(p,t) \leq r^{-1}$, then*

$$\sup_{P(p,t,\rho r)} H \leq C r^{-1}\,. \tag{3.10}$$

For comparison, recall that if u is a positive solution of an elliptic or parabolic partial differential equation, then by the classical Harnack estimate the values of u at nearby points are comparable.

Proof. Suppose towards a contradiction that the assertion fails. Then, there is a sequence $\mathcal{M}^j$ of α-noncollapsed flows defined in $P(0,0,j)$ with $H(0,0) \leq j^{-1}$, but such that

$$\sup_{P(0,0,1)} H \geq j. \tag{3.11}$$

We can assume that the outward normal of M_0^j at the origin is e_3, and that for every $R < \infty$ the flows $\mathcal{M}^j$ foliate $B(0,R)$ for $j \geq j_0(R)$.

We claim that the sequence $\mathcal{M}^j$ converges in the pointed Hausdorff sense to the static plane $\{x_3 = 0\}$ in $\mathbb{R}^3 \times (-\infty, 0]$. Indeed, for any $R < \infty$ and $d > 0$ consider the closed ball $B_{R,d} \subset \{x_3 \leq d\}$ of radius R that touches the point de_3. When R is large, it will take approximately time Rd for $B_{R,d}$ to leave the upper halfspace $\{x_3 > 0\}$. Since $0 \in M_0^j$ for all j, it follows that $B_{R,d}$ cannot be contained in the interior of M_t^j for any $t \in [-T, 0]$, where $T \simeq Rd$. Thus, for large j we can find $d_j \leq d$ such that B_{R,d_j} has interior contact with M_t^j at some point q_j, where $\langle q_j, e_3\rangle < d$, $\|q_j\| \lesssim \sqrt{Rd}$, and $\liminf_{j\to\infty}\langle q_j, e_3\rangle \geq 0$. Now, since M_t^j satisfies the α-noncollapsing condition, there is a closed ball B_j with radius at least $\alpha R/2$ making exterior contact with M_0^j at q_j. By a simple geometric calculation, this implies that M_t^j has height $\lesssim d/\alpha$ in the ball $B(0, R')$, where R' is comparable to $\sqrt{Rd}$. As d and R are arbitrary, this implies that for any $T > 0$, and any compact subset $Y \subset \{x_3 > 0\}$, for large j the time slice M_t^j is disjoint from Y, for all $t \geq -T$. Finally, observe that for any $T > 0$ and any compact subset $Y \subset \{x_3 < 0\}$, the time slice M_t^j contains Y for all $t \in [-T, 0]$, and large j, because M^j_{-T} contains a ball whose forward evolution under the flow contains Y at any time $t \in [-T, 0]$. This proves the claim.

Finishing the proof of the theorem, together with one-sided minimization (see below), we infer that for every $\varepsilon > 0$, every $t \leq 0$ and every ball $B(x, r)$ centered on the hyperplane $\{x_3 = 0\}$ we have

$$\mathrm{Area}(M^j_t \cap B(x, r)) \leq (1 + \varepsilon)\pi r^2 \,, \tag{3.12}$$

whenever j is large enough. Finally, applying the epsilon-regularity theorem for the mean curvature flow (Theorem 2.14) this yields

$$\limsup_{j\to\infty} \sup_{P(0,0,1)} |A| = 0 \,. \tag{3.13}$$

This contradicts (3.11), and thus concludes the proof. □

Exercise 3.14 (One-sided minimization)**.** *Use Stokes' theorem and mean-convexity to prove the density bound* (3.12)*.*

The next estimate gives pinching of the curvatures towards positive:

Theorem 3.15 (Convexity estimate)**.** *For all $\varepsilon > 0$ and $\alpha > 0$, there exists a constant $\eta = \eta(\varepsilon, \alpha) < \infty$ with the following significance. If $\mathcal{M}$ is an α-noncollapsed flow defined in a parabolic ball $P(p, t, \eta r)$ centered at a point $p \in M_t$ with $H(p, t) \leq r^{-1}$, then*

$$\kappa_1(p, t) \geq -\varepsilon r^{-1}. \tag{3.16}$$

In particular, any ancient α-noncollapsed flow $\{M_t \subset \mathbb{R}^3\}_{t\in(-\infty,T)}$, for example any blowup limit of an α-noncollapsed flow, is convex.

Proof. Fixing α, let $\varepsilon_0 \leq 1/\alpha$ be the infimum of the epsilons for which the assertion holds, and suppose towards a contradiction that $\varepsilon_0 > 0$. Then, there is a sequence $\mathcal{M}^j$ of α-noncollapsed flows defined in $P(0, 0, j)$, such that $(0, 0) \in \mathcal{M}^j$ and $H(0, 0) \leq 1$, but $\kappa_1(0, 0) \to -\varepsilon_0$ as $j \to \infty$. By Theorem 3.9 (local curvature estimate), after passing to a subsequence, $\mathcal{M}^j$ converges smoothly to a limit $\mathcal{M}^\infty$ in $P(0, 0, \rho/2)$. Observe that for $\mathcal{M}^\infty$ we have $\kappa_1(0, 0) = -\varepsilon_0$ and thus $H(0, 0) = 1$.

By continuity $H > 1/2$ in $P(0, 0, r)$ for some $r \in (0, \rho/2)$. Furthermore, we have $\kappa_1/H \geq -\varepsilon_0$ everywhere in $P(0, 0, r)$. This is because every $(p, t) \in \mathcal{M}^\infty \cap P(0, 0, r)$ is a limit of points $(p_j, t_j) \in \mathcal{M}^j$, and for every $\varepsilon > \varepsilon_0$, if $\eta = \eta(\varepsilon, \alpha)$, then for large j enough $\mathcal{M}^j$ is defined in $P(p_j, t_j, \eta/H(p_j, t_j))$, which implies that $\kappa_1(p_j, t_j) \geq -\varepsilon H(p_j, t_j)$. Thus, in $P(0, 0, r)$ the ratio κ_1/H attains a negative minimum $-\varepsilon_0$ at $(0, 0)$. Since $\kappa_1 < 0$ and $H > 0$ the Gauss curvature $K = \kappa_1\kappa_2$ at the origin is strictly negative. However, by the equality case of the maximum principle for κ_1/H, the surface locally splits as a product and thus the Gauss curvature must vanish; a contradiction. □

Pushing the above methods a bit further, via an induction on scale argument it can be shown that all blowup limits of α-noncollapsed flows are smooth and convex until they become extinct. Together with the recent classification by Brendle-Choi, it then follows that for the flow of mean-convex embedded surfaces all singularities at the first singular time are modelled either by a round shrinking sphere, a round shrinking cylinder or a self-similarly translating bowl soliton. This makes precise the intuition that, unless the entire surface shrinks to a round point, all singularities look like neck-pinches or degenerate neck-pinches.

Related PDEs. The notion of α-noncollapsing for the mean curvature flow is inspired by Perelman's κ-noncollapsing for the Ricci flow [Per02]. For example, α-noncollapsing rules out blowup limits like $\mathbb{R}\times$Grim-Reaper, and κ-noncollapsing rules out blowup limits like $\mathbb{R}\times$Cigar. More generally, conditions with touching balls are used frequently to establish estimates for elliptic or parabolic PDEs. Also, as discussed, the curvature estimate is related, at least in spirit, to Harnack inequalities for positive solutions of elliptic or parabolic PDEs. Finally, for 3d Ricci flow there is the Hamilton-Ivey convexity estimate [Ham95].

References. Noncollapsing for mean-convex mean curvature flow was proved first by White [Whi00]. The notion of α-noncollapsing was introduced by Sheng-Wang [SW09], and the beautiful maximum principle proof that it is preserved is due to Andrews [And12], see also [Bre14, Has14] for expositions. The theory of mean-convex mean curvature flow has been established first in the fundamental work of White [Whi00, Whi03] and Huisken-Sinestrari [HS99a, HS99b]. The streamlined treatment in the setting of α-noncollapsed flows presented here is from my joint work with Kleiner [HK17]. Finally, ancient α-noncollapsed flows of surfaces have been classified in a recent breakthrough by Brendle-Choi [BC19] and Angenent-Daskalopoulos-Sesum [ADS20].

4. Weak solutions

In this lecture, we discuss notions of weak (aka generalized) solutions that allow one to continue the evolution through singularities.

Recall that by the avoidance principle smooth mean curvature flows do not bump into each other. Motivated by this, a family of closed sets $\{C_t\}$ is called a *subsolution* if it avoids all smooth solutions, namely

$$C_{t_0} \cap M_{t_0} = \emptyset \qquad \Rightarrow \qquad C_t \cap M_t = \emptyset \quad \forall t \in [t_0, t_1]\,, \tag{4.1}$$

whenever $\{M_t\}_{t\in[t_0,t_1]}$ is a smooth mean curvature flow of closed surfaces.

Definition 4.2 (level-set flow)**.** The *level-set flow* $\{F_t(C)\}_{t\geq 0}$ of any closed set C is the maximal subsolution $\{C_t\}_{t\geq 0}$ with $C_0 = C$.

Proposition 4.3 (basic properties)**.** *The level-set flow is well-defined and unique, and has the following basic properties:*

- *semigroup property:* $F_0(C) = C$, $F_{t+t'}(C) = F_t(F_{t'}(C))$.
- *commutes with translations:* $F_t(C + x) = F_t(C) + x$.
- *containment: if* $C \subseteq C'$, *then* $F_t(C) \subseteq F_t(C')$.

Proof. Observe first that by translation-invariance of smooth solutions, a family of closed sets $\{C_t\}$ is a subsolution if and only if

$$d(C_t, M_t) \geq d(C_{t_0}, M_{t_0}) \quad \forall t \in [t_0, t_1]\,, \tag{4.4}$$

whenever $\{M_t\}_{t\in[t_0,t_1]}$ is a smooth closed mean curvature flow. Now, considering the closure of the union of all subsolutions, namely

$$F_{t'}(C) = \overline{\bigcup \{C_{t'} \mid \{C_t\}_{t\geq 0} \text{ is a subsolution}\}}\,, \tag{4.5}$$

we see that the level-set flow exists and is unique. Finally, the basic properties immediately follow from existence and uniqueness. □

Using the characterization (4.4) it is also not hard to see that level-set solutions are consistent with classical solutions, namely if $\{M\}_{t\in[0,T)}$ is a smooth mean curvature flow of closed surfaces, then

$$F_t(M) = M_t \qquad \forall t \in [0, T). \tag{4.6}$$

Furthermore, by interposing a $C^{1,1}$-surface one can check that level-set flows also avoid each other, namely

$$C \cap C' = \emptyset \qquad \Rightarrow \qquad F_t(C) \cap F_t(C') = \emptyset \quad \forall t \geq 0\,, \tag{4.7}$$

provided that at least one of C, C' is compact. While the level-set solution is unique by definition, the evolution can be nonunique:

Example 4.8 (fattening)**.** There exists a closed embedded surface $M \subset \mathbb{R}^3$, which looks like a wheel with many spokes, that encounters a conical singularity after which $F_t(M)$ develops nonempty interior.

For the sake of intuition, it helps to compare this with the nonuniqueness/fattening of the figure X under curve shortening flow. Now, to capture this nonuniqueness phenomenon in more detail, given any closed embedded surface $M \subset \mathbb{R}^3$ we denote by K the compact domain enclosed by M, and set $K' := \overline{K^c}$. Note that $\partial K = M = \partial K'$. We then consider the space-time tracks of their level-set flows, namely

$$\mathcal{K} := \{(x,t) \in \mathbb{R}^3 \times [0,\infty) \,|\, x \in F_t(K)\}\,, \tag{4.9}$$

and

$$\mathcal{K}' := \{(x,t) \in \mathbb{R}^3 \times [0,\infty) \,|\, x \in F_t(K')\}\,. \tag{4.10}$$

Definition 4.11 (outer and inner flow)**.** The *outer flow* is define by

$$M_t := \{x \in \mathbb{R}^3 \,|\, (x,t) \in \partial\mathcal{K}\}\,, \tag{4.12}$$

and the *inner flow* is defined by

$$M_t' := \{x \in \mathbb{R}^3 \,|\, (x,t) \in \partial\mathcal{K}'\}\,. \tag{4.13}$$

Here, for technical reasons it is most convenient to work with the boundary of space-time sets, but alternatively one can check that

$$M_t = \lim_{t' \nearrow t} \partial F_{t'}(K)\,, \tag{4.14}$$

and similarly for the inner flow.

Definition 4.15 (discrepancy time)**.** The *discrepancy time* is

$$T_{\mathrm{disc}} := \inf\{\, t > 0 \,|\, M_t \neq M_t' \,\} \in (0,\infty]\,. \tag{4.16}$$

This captures the first time when nonuniqueness happens.[4]

While level-set solutions are very well suited for discussing the question of uniqueness versus nonuniqueness, we also need another notion of solutions, so called Brakke flows, that is better suited for arguments based on the monotonicity formula and for passing to limits.

Recall that a Radon measure μ in $\mathbb{R}^3$ is integer two-recifiable, if at almost every point it possess a tangent plane of integer multiplicity. Namely, setting $\mu_{x,\lambda}(A) = \lambda^{-2}\mu(\lambda A + x)$, for μ-a.e. x we have

$$\lim_{\lambda \to 0} \mu_{x,\lambda} = \theta \mathcal{H}^2 \lfloor P\,, \tag{4.17}$$

[4]The discrepancy time T_{disc} is always less than or equal to the fattening time $T_{\mathrm{fat}} := \inf\{\, t > 0 \,|\, \mathrm{Int}(F_t(M)) \neq 0\}$. It is an open problem to show that they are equal, but in any case discrepancy is the better notion to capture nonuniqueness.

for some positive integer θ and some plane P. We write $P = T_x\mu$. Also recall that the associated integral varifold is defined by

$$V_\mu(\psi) = \int \psi(x, T_x\mu)\, d\mu(x)\,. \tag{4.18}$$

Definition 4.19 (Brakke flows)**.** A two-dimensional *integral Brakke flow* in $\mathbb{R}^3$ is a family of Radon measures $\mathcal{M} = \{\mu_t\}_{t\in I}$ in $\mathbb{R}^3$ that is integer two-rectifiable for almost every time and satisfies

$$\frac{d}{dt}\int \varphi\, d\mu_t \leq \int \left(-\varphi \vec{H}^2 + D\varphi\cdot\vec{H}\right)\, d\mu_t \tag{4.20}$$

for all test functions $\varphi\in C^1_c(\mathbb{R}^3,\mathbb{R}_+)$. Here, $\frac{d}{dt}$ denotes the limsup of difference quotients, and $\vec{H}$ denotes the mean curvature vector of the associated varifold V_{μ_t}.[5]

The definition is of course motivated by fact that for smooth solutions (4.20) would hold as equality. In general though, only the inequality is preserved under passing to weak limits. All integral Brakke flows that we encounter in this lecture series are:

- *unit-regular*, i.e. near every space-time point of Gaussian density 1 the flow is regular in a two-sided parabolic ball, and
- *cyclic*, i.e. for a.e. t the $\mathbb{Z}_2$ flat chain $[V_{\mu_t}]$ satisfies $\partial[V_{\mu_t}] = 0$.

Intuitively, the last item simply means that we can color the inside and outside, which in particular rules out blowup limits like $\mathrm{Y}\times\mathbb{R}$.

Also, if M_t is any smooth mean curvature flow, then $\mu_t := \mathcal{H}^2\lfloor M_t$ is of course a unit-regular, cyclic, integral Brakke flow.

Theorem 4.21 (compactness)**.** *Any sequence of integral Brakke flows μ^i_t with uniform area bounds on compact subsets has a subsequence $\mu^{i'}_t$ that converges to an integral Brakke flow μ_t, namely (i) for every t we have $\mu^{i'}_t \to \mu_t$ as Radon measures, and (ii) for a.e. t after passing to a further subsequence $i'' = i''(t)$ we have $V_{\mu^{i''}_t} \to V_{\mu_t}$ as varifolds. Moreover, if the sequence is unit-regular/cyclic, then so is the limit.*

Proof (sketch). Using (4.20) and the Peter-Paul inequality

$$D\varphi\cdot\vec{H} \leq \frac{1}{2}\frac{|D\varphi|^2}{\varphi} + \frac{1}{2}\varphi|\vec{H}|^2\ , \tag{4.22}$$

[5]By convention, the right hand side is interpreted as $-\infty$ whenever it does not make sense literally. Hence, it actually makes sense literally at almost every time.

we see that for every $\varphi \in C^2_c(\mathbb{R}^3;\mathbb{R}_+)$ there exists $C(\varphi)<\infty$ such that

$$L^i_\varphi(t) := \int \varphi d\mu^i_t - C(\varphi)t \tag{4.23}$$

is decreasing in t. Hence, after passing to a subsequence, that may depend on φ, the functions $L^i_\varphi(t)$ converge to a monotone function $L(t)$. In particular, $\int \varphi d\mu^i_t$ has a limit for all t. Repeating this process for a countable dense subset of $C^2_c(\mathbb{R}^3;\mathbb{R}_+)$ we can arrange that

$$\mu^i_t \to \mu_t \tag{4.24}$$

as Radon measures for all t. Now, by the assumed area bounds, and the inequalities (4.20) and (4.22) we have

$$\sup_{t\in[t_0,t_1]} \mu_t(K) + \int_{t_0}^{t_1}\int_K H^2\, d\mu^i_t\, dt \leq C(K,t_1,t_2) \tag{4.25}$$

for every compact set $K\subset\mathbb{R}^3$. Hence, by Allard's compactness theorem, for a.e. t we can find a subsequence $i(t)$ such that

$$V_{\mu^{i(t)}_t} \to V_{\mu_t} \tag{4.26}$$

as integral varifolds. Furthermore, by Fatou's lemma we have

$$\int \varphi|\vec{H}|^2\, d\mu_t \leq \liminf_{i\to\infty}\int \varphi|\vec{H}_i|^2\, d\mu^i_t\,. \tag{4.27}$$

Hence, Brakke's inequality (4.20) passes to the limit. Moreover, by Theorem 2.14 (epsilon-regularity) the convergence is smooth near points with Gaussian density close to 1, so being unit-regular is preserved. Likewise, using again (4.25) it follows from a result of White that

$$[V_{\mu^{i(t)}_t}] \to [V_{\mu_t}] \tag{4.28}$$

as mod 2 flat chains, so being cyclic is also preserved. □

Furthermore, plugging the backwards heat kernel times a cutoff function in the definition of Brakke flow it is not hard to check that the computation from the second lecture goes through, yielding

$$\frac{d}{dt}\int \rho_{X_0}\chi^\rho_{X_0} d\mu_t \leq -\int \left|\vec{H} - \frac{(x-x_0)^\perp}{2(t-t_0)}\right|^2 \rho_{X_0}\chi^\rho_{X_0} d\mu_t\,. \tag{4.29}$$

Finally, the notions of outer/inner flow and Brakke flow are compatible. Specifically, given any closed embedded initial surface M_0 using Ilmanen's elliptic regularization one can construct a unit-regular, cyclic, integral Brakke flow μ_t and μ'_t with initial condition $\mathcal{H}^2\lfloor M_0$ such that the support is given by the closed sets M_t and M'_t, respectively.

Related PDEs. Weak solutions for the harmonic map flow that satisfy the monotonicity inequality have been constructed by Chen-Struwe [CS89]. Their proof shares some similarities with the construction of weak solutions for the Navier-Stokes equation by Leray [Ler34]. Constructing weak solutions for the Ricci flow is still a major open problem, for recent progress in this direction see [KL17, HN18, Stu18, Bam20].

References. The level-set flow was first studied numerically by Osher-Sethian [OS88], and in the rigorous setting of viscosity solutions by Chen-Giga-Goto [CGG91] and Evans-Spruck [ES91]. The geometric reformulation presented here is due to Ilmanen [Ilm93], see also [HW18]. The fattening example with smooth closed initial condition is due to Ilmanen-White [Whi02]. The notion of discrepancy was introduced by Hershkovits-White [HW20]. Evolving varifolds were introduced by Brakke [Bra78]. The compactness theorem for integral Brakke flows and existence via elliptic regularization can be found in Ilmanen's monograph [Ilm94], see also [HW20]. Finally, for the notions unit-regular and cyclic, and their preservation, see [SW20] and [Whi09].

5. Flow through neck-singularities

In this lecture, we discuss our recent proof of the mean-convex neighborhood conjecture and some of its consequences.

As we have seen, one the one hand there is a very well-developed theory in the mean-convex case, but on the other hand, the general case without any assumption on the sign of the mean curvature is much less understood. The main conjecture towards decreasing this gap of understanding is Ilmanen's mean-convex neighborhood conjecture:

Conjecture 5.1 (mean-convex neighborhoods)**.** *If $\mathcal{M} = \{M_t\}_{t\geq 0}$ has a neck-singularity at (x_0, t_0), then there exists $\varepsilon = \varepsilon(x_0, t_0) > 0$ such that $M_t \cap B_\varepsilon(x_0)$ is mean-convex for $|t - t_0| < \varepsilon$.*

Let us explain the statement. Given any space-time point $X_0 = (x_0, t_0)$ we consider a tangent-flow at X_0, namely

$$\hat{\mathcal{M}}_{X_0} := \lim_{i'\to\infty} \mathcal{D}_{\lambda_i}(\mathcal{M} - X_0)\ , \tag{5.2}$$

i.e. we shift X_0 to the space-time origin, parabolically dilate by $\lambda_i \to \infty$, and pass to a subsequential limit. By Husiken's monotonicity formula and the compactness theorem for integral Brakke flows, tangent-flows always exit and are always self-similarly shrinking. The assumption that $\mathcal{M}$ has a neck-singularity at X_0 means that

$$\hat{\mathcal{M}}_{X_0} = \{S^1(\sqrt{2|t|}) \times \mathbb{R}\}_{t<0}\,, \tag{5.3}$$

for some (and thus any) choice of rescaling factors $\lambda_i \to \infty$.

In joint work with Choi and Hershkovits we proved the conjecture:

Theorem 5.4 (mean-convex neighborhoods). *If $\mathcal{M} = \{M_t\}_{t\geq 0}$ has a neck-singularity at (x_0, t_0), then there exists $\varepsilon = \varepsilon(x_0, t_0) > 0$ such that $M_t \cap B_\varepsilon(x_0)$ is mean-convex for $|t - t_0| < \varepsilon$.*

A major difficulty was to rule out the potential scenario of a degenerate neck-pinch with a non-convex cap. The problem is that tangent-flows only partially capture singularities and in particular do not detect the cap region. To fully capture the singularity, one really wants to understand all limit flows,

$$\mathcal{M}^\infty := \lim_{\lambda_i \to \infty} \mathcal{D}_{\lambda_i}(\mathcal{M} - X_i)\,, \tag{5.5}$$

where now $X_i \to X_0$ depends on i (e.g. for the degenerate neck-pinch one chooses X_i along the tip). While tangent-flows are always self-similarly shrinking by Huisken's monotonicity formula, limit flows can be much more general. A priori, choosing $\lambda_i \to \infty$ suitably, we only know that $\mathcal{M}^\infty$ is an ancient asymptotically cylindrical flow:

Definition 5.6. An *ancient asymptotically cylindrical flow* is an ancient, unit-regular, cyclic, integral Brakke flow $\mathcal{M} = \{\mu_t\}_{t\in(-\infty,T_E)}$ whose tangent flow at $-\infty$ is a round shrinking cylinder, namely

$$\check{\mathcal{M}} := \lim_{\lambda_i \to 0} \mathcal{D}_{\lambda_i}\mathcal{M} = \{S^1(\sqrt{2|t|}) \times \mathbb{R}\}_{t<0}\,. \tag{5.7}$$

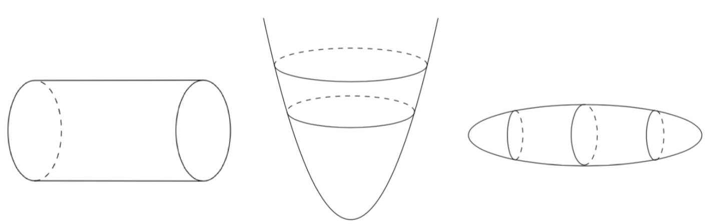

FIGURE 1. Cylinder, bowl and ancient oval.

Together with Choi and Hershkovits we classified all such flows:

Theorem 5.8 (classification). *Any ancient asymptotically cylindrical flow in $\mathbb{R}^3$ is one of the following:*

- *round shrinking cylinder*
- *translating bowl*
- *ancient oval.*

The classification is illustrated in Figure 1. In particular, note that all solutions appearing in the classification result are convex. Hence, the mean-convex neighborhood conjecture follows from the classification theorem via a short argument by contradiction.

Our theorem generalizes prior classification results by Wang, myself, Bernstein-Wang, Brendle-Choi and Angenent-Daskalopoulos-Sesum. In stark contrast to all prior results, the major improvement is that we assume neither self-similarity nor convexity. In fact, this seems to be the first instance for any geometric flow, where a classification in such a general setting has been accomplished, and this is of course absolutely crucial for the proof of the mean-convex neighborhood conjecture.

As an application we proved the uniqueness conjecture for mean curvature flow through neck-singularities:

Theorem 5.9 (uniqueness). *Mean curvature flow through neck singularities is unique.*

As another application we proved White's sphere conjecture conditional on Ilmanen's multiplicity-one conjecture:

Theorem 5.10 (sphere). *Assuming Ilmanen's multiplicity-one conjecture, mean curvature flow of embedded 2-spheres is well posed.*

Let us explain how these results follow. It has been known since the 90s that mean-convex flows are unique. More recently, Hershkovits-White localized this result and showed that it is enough to assume that all singularities have a mean-convex neighborhood, and we exactly established this assumption. For the proof of the sphere-conjecture we use in addition that by a result of Brendle the only nontrivial shrinkers of genus zero are the round sphere and the round cylinder.

To conclude this lecture, let us outline the main steps of our proof of the classification theorem. Given any ancient asymptotically cylindrical flow $\mathcal{M}$ that is not a round shrinking cylinder we have to show that it is either a bowl or an oval.

To get started, we set up a fine-neck analysis as follows. Given any $X_0 = (x_0, t_0)$ we consider the renormalized flow

$$\bar{M}_\tau^{X_0} = e^{\tau/2}(M_{t_0 - e^{-\tau}} - x_0)\,. \tag{5.11}$$

Then, $\bar{M}_\tau^{X_0}$ converges for $\tau \to -\infty$ to $S^1(\sqrt{2}) \times \mathbb{R}$. Hence, writing $\bar{M}_\tau^{X_0}$ locally as a graph of a function $u^{X_0}(z, \theta, \tau)$ over the cylinder the evolution is governed by the Ornstein-Uhlenbeck type operator

$$\mathcal{L} = \partial_z^2 - \tfrac{1}{2} z \partial_z + \tfrac{1}{2}\partial_\theta^2 + 1\,. \tag{5.12}$$

This operator has 4 unstable eigenfunctions, namely 1, $\sin\theta$, $\cos\theta$, z, and 3 neutral eigenfunctions, namely $z\sin\theta$, $z\cos\theta$, $z^2 - 2$, and all other eigenfunctions are stable. By the Merle-Zaag ODE lemma for $\tau \to -\infty$ either the unstable or neutral eigenfunctions dominate.

If the unstable-mode is dominant, we prove that there exists a constant $a = a(\mathcal{M}) \neq 0$ independent of the center point X_0 such that (after suitable recentering to kill the rotations) we have

$$u^{X_0}(z, \theta, \tau) = aze^{\tau/2} + o(e^{\tau/2}) \tag{5.13}$$

for all $\tau \ll 0$ depending only on the cylindrical scale of X_0. Moreover, we show that every point outside a ball of controlled size in fact lies on such a fine-neck. Hence, by the Brendle-Choi neck-improvement theorem the solution becomes very symmetric at infinity. Finally, we can apply a variant of the moving plane method to conclude that the solution is smooth and rotationally symmetric, and hence the bowl.

If the neutral-mode is dominant, then analyzing to ODE for the coefficient of the eigenfunction z^2-2 we show that there is an inwards quadratic bending, and consequently that the solution is compact. Blowing up near the tips, by the classification from the unstable-mode case we see bowls. Hence, using the maximum principle we can show that the flow is mean-convex and noncollapsed. Finally, we can apply the result by Angenent-Daskalopoulos-Sesum to conclude that it is an oval.

Related PDEs. Bamler-Kleiner recently proved uniqueness of 3d Ricci flow through singularities [BK17]. However, for 3d Ricci flow thanks to the Hamilton-Ivey pinching estimate one knows a priori that all blowup limits are convex, so this is morally most comparable to the flow of mean-convex surfaces. Motivated by our proof of the mean-convex neighborhood conjecture, it seems likely that there should be a canonical neighborhood theorem for neck-singularities in higher dimensional Ricci flow without assuming convexity a priori.

References. The results presented in this section are from my joint paper with Choi and Hershkovits [CHH18]. The reformulation in terms of ancient asymptotically cylindrical flows and the variant of the moving plane method is from our follow up paper [CHHW19]. The important prior classification results that we discussed can be found in [Wan11, Has15, BW17, BC19, ADS20]. The Merle-Zaag ODE lemma is from [MZ98]. Brendle's classification of genus zero shrinkers appeared in [Bre16], and the Brendle-Choi neck-improvement in [BC19]. Yet another important ingredient is the uniqueness of cylindrical tangent-flows via the Lojasiewicz inequality from Colding-Minicozzi [CM15].

6. Open problems

In this final section, we discuss some of the most important open problems for the mean curvature flow of surfaces.

Conjecture 6.1 (Multiplicity one-conjecture). *If the initial surface is closed and embedded, then all blowup limits have multiplicity one.*

This is presumably the biggest open problem, ever since Brakke's pioneering work. Here, to be specific one can assume that the flow is constructed via Ilmanen's elliptic regularization procedure (and thus is a unit-regular, cyclic, integral Brakke flow). In particular, one has to rule out singularities that locally look like two planes connected by small tubes and have a multiplicity-two plane as a blowup limit.

Conjecture 6.2 (No cylinder conjecture). *The only complete embedded shrinker with a cylindrical end is the round cylinder.*[6]

To address this conjecture, one in particular has to rule out the scenario of shrinkers of mixed-type where some ends are cylindrical and some ends are conical. The statement becomes false if the completeness or embeddedness assumption is dropped. Together with prior work of Wang, a resolution of the conjecture would imply the nice dichotomy that all singularities are either of conical-type or of neck-type.

Conjecture 6.3 (Uniqueness of tangent-flows). *Tangent-flows are independent of the choice of sequence of rescaling factors $\lambda_i \to \infty$.*

[6]I do not have any convincing heuristics why this should be true, so to err on the side of caution it might be better to call it a question instead of a conjecture.

By results of Schulze, Colding-Minicozzi and Chodosh-Schulze this would follow from a resolution of the multiplicity-one conjecture and the no cylinder conjecture. Alternatively, one may also try to establish uniqueness directly without relying on the no cylinder conjecture.

Conjecture 6.4 (Bounded diameter conjecture)**.** *The intrinsic diameter stays uniformly bounded as one approaches the first singular time.*

This is motivated by a corresponding conjecture of Perelman for 3d Ricci flow. Du proved that the bounded diameter conjecture would follow from a resolution of the multiplicity-one conjecture and the no cylinder conjecture. Alternatively, one may also try to establish the diameter bound directly without relying on the no cylinder conjecture.

Conjecture 6.5 (Genericity conjecture)**.** *For generic initial data all singularities are of neck-type or spherical-type.*

Here, generic of course means open and dense. Openness for flows that only encounter multiplicity-one neck and spherical singularities follows from our resolution of the mean-convex neighborhood conjecture. Regarding denseness, Chodosh-Choi-Mantoulidis-Schulze recently proved that this would follow from a resolution of the multiplicity-one conjecture and the no cylinder conjecture. Alternatively, instead of trying to rule out potential counterexamples to the no cylinder conjecture, one may try directly to perturb them away.

Next, let me discuss several conjectures about the size and the structure of the singular set. To be specific one can again assume that the flow through singularities is constructed via Ilmanen's elliptic regularization procedure (and thus is unit-regular, cyclic, and integral).

Conjecture 6.6 (Partial regularity conjecture)**.** *If the initial surface is closed and embedded, then the parabolic Hausdorff dimension of the singular set is at most* 1*.*

By standard dimension reduction, this would follow from a resolution of the multiplicity-one conjecture. Alternatively, one may also try to address partial regularity directly. A stronger form would be:

Conjecture 6.7 (Isolation conjecture)**.** *All singularities are isolated unless an entire tube shrinks to a closed curve.*

The conjecture is motivated by the principle that solutions of the level set flow, while only twice-differentiable, to some extent behave like analytic functions. A somewhat weaker formulation is:

Conjecture 6.8 (Finiteness of singular times)**.** *There are only finitely many singular times.*

Another well-known related open problem is:

Open problem 6.9 (Self-similarity of blowup limits)**.** *Are blowup limits always selfsimilar?*

In recent work with B. Choi and Hershkovits we proved that the ancient ovals occur as blowup limit if and only if there is an accumulation of spherical singularities. Yet another related open problem is:

Open problem 6.10 (Shrinking tubes)**.** *Which closed curves can arise as singular set of a mean curvature flow of embedded surfaces?*

The only known example is the marriage-ring, which shrinks to a round circle.

Let us now switch gears and discuss some problems that specifically assume that the surface is topologically a two-sphere:

Conjecture 6.11 (Two-sphere conjecture)**.** *Mean curvature flow of embedded two-spheres is well-posed.*

By our resolution of the mean-convex neighborhood conjecture and Brendle's classification of genus zero shrinkers, this would follow from a resolution of the multiplicity-one conjecture.

Open problem 6.12 (Mean curvature flow proof of Smale conjecture)**.** *Is there a mean curvature flow proof of the Smale conjecture?*

In one of its many formulations, Smale's conjecture states that the space of embedded two-spheres in $\mathbb{R}^3$ is contractible. There is a geometric proof by Hatcher and a Ricci flow proof by Bamler-Kleiner, but it would be nice to have a direct proof by mean curvature flow. Likely, a resolution of the two-sphere conjecture, would yield such a proof. Another classical problem related to the two-sphere conjecture is:

Conjecture 6.13 (Lusternik-Schnirelman type conjecture)**.** *The three-sphere equipped with any Riemannian metric contains at least* 4 *embedded minimal two-spheres.*

This is motivated by a classical result of Lusternik-Schnirelman and Grayson that establishes the existence of at least 3 closed embedded geodesics on the two-sphere with any metric. Simon-Smith proved that there always is at least 1 embedded minimal two-sphere, and more recently with Ketover we found a second one for generic metrics.

Finally, let me mention two somewhat more open ended problems:

Open problem 6.14 (Selection principle)**.** *Is there a selection principle for flowing out of conical singularities?*

It has been proposed by Dirr-Luckhaus-Novaga and Yip that considering mean curvature flow with space-time white noise the physically relevant solutions will be selected in the vanishing noise limit.

Open problem 6.15 (Immersed surfaces)**.** *Develop a theory of weak solutions for the flow of immersed surfaces.*

Many of the methods described in this lecture series rely on embeddedness, so some fundamentally new ideas would be needed.

Related PDEs. Many of the problems discussed here, including in particular the uniqueness of tangent-cones/flows, genericity, optimal partial regularity, finiteness of singularities, self-similarity of blowup limits and selection principle for evolution through singularities, are of central importance in a wide range of partial differential equations.

References. Many of the questions discussed here can be found on Ilmanen's problem list [Ilm03]. Multiplicity-one first appeared as an hypothesis in Brakke's work [Bra78], and then has been upgraded to a conjecture by Ilmanen. Some recent progress under additional assumptions has been made in [Sun18, LW18], though the general case remains widely open. Wang proved that the ends of shrinkers are always either conical or cylindrical [Wan16]. Uniqueness of compact, cylindrical and asymptotically conical tangent-flows has been established in [Sch14, CM15, CS19]. Special cases of the bounded diameter conjecture have been proved in [GH20, Du20]. The genericity conjecture is due to Huisken, and some exciting progress has been made by Colding-Minicozzi [CM12] and Chodosh-Choi-Mantoulidis-Schulze [CCMS20]. The principle that solutions of the level set flow to some extent behave like analytic functions is due to Colding-Minicozzi [CM16]. The potential scenario of ancient ovals as limit flows has been investigated in [CHH21]. The two-sphere conjecture has been stated in White's ICM-lecture [Whi02]. Brendle proved that the only nontrivial genus zero shrinkers are the round sphere and the round cylinder [Bre16]. The references for the proofs of the Smale conjecture are [Hat83, BK19].

The min-max construction by Simon-Smith is from [Smi82], and the existence of a second minimal two-sphere has been proved in my joint work with Ketover [HK19]. Finally, the stochastic selection principle has been proposed in [DLN01, Yip98].

References

[ADS20] S. Angenent, P. Daskalopoulos, and N. Sesum. Uniqueness of two-convex closed ancient solutions to the mean curvature flow. *Ann. of Math. (2)*, 192(2):353–436, 2020.

[All72] W. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.

[And90] M. Anderson. Convergence and rigidity of manifolds under Ricci curvature bounds. *Invent. Math.*, 102(2):429–445, 1990.

[And12] B. Andrews. Noncollapsing in mean-convex mean curvature flow. *Geom. Topol.*, 16(3):1413–1418, 2012.

[Bam20] R. Bamler. Compactness theory of the space of Super Ricci flows. *arXiv:2008.09298*, 2020.

[BC19] S. Brendle and K. Choi. Uniqueness of convex ancient solutions to mean curvature flow in $\mathbb{R}^3$. *Invent. Math.*, 217(1):35–76, 2019.

[BHH] R. Buzano, R. Haslhofer, and O. Hershkovits. The moduli space of two-convex embedded spheres. *J. Differential. Geom. (to appear)*.

[BK17] R. Bamler and B. Kleiner. Uniqueness and stability of Ricci flow through singularities. *arXiv:1709.04122*, 2017.

[BK19] R. Bamler and B. Kleiner. Ricci flow and contractibility of spaces of metrics. *arXiv:1909.08710*, 2019.

[Bra78] K. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1978.

[Bre14] S. Brendle. Two-point functions and their applications in geometry. *Bull. Amer. Math. Soc. (N.S.)*, 51(4):581–596, 2014.

[Bre16] S. Brendle. Embedded self-similar shrinkers of genus 0. *Ann. of Math. (2)*, 183(2):715–728, 2016.

[BW17] J. Bernstein and L. Wang. A topological property of asymptotically conical self-shrinkers of small entropy. *Duke Math. J.*, 166(3):403–435, 2017.

[CCMS20] O. Chodosh, K. Choi, C. Mantoulidis, and F. Schulze. Mean curvature flow with generic initial data. *arXiv:2003.14344*, 2020.

[CGG91] Y.G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[CHH18] K. Choi, R. Haslhofer, and O. Hershkovits. Ancient low entropy flows, mean convex neighborhoods, and uniqueness. *arXiv:1810.08467*, 2018.

[CHH21] B. Choi, R. Haslhofer, and O. Hershkovits. A note on the selfsimilarity of limit flows. *Proc. Amer. Math. Soc.*, 149(3):1239–1245, 2021.

[CHHW19] K. Choi, R. Haslhofer, O. Hershkovits, and B. White. Ancient asymptotically cylindrical flows and applications. *arXiv:1910.00639*, 2019.

[CM12] T. Colding and W. Minicozzi. Generic mean curvature flow I; generic singularities. *Ann. of Math. (2)*, 175(2):755–833, 2012.

[CM15] T. Colding and W. Minicozzi. Uniqueness of blowups and Lojasiewicz inequalities. *Ann. of Math. (2)*, 182(1):221–285, 2015.

[CM16] T. Colding and W. Minicozzi. Differentiability of the arrival time. *Comm. Pure Appl. Math.*, 69(12):2349–2363, 2016.

[CS89] Y. Chen and M. Struwe. Existence and partial regularity results for the heat flow for harmonic maps. *Math. Z.*, 201(1):83–103, 1989.

[CS19] O. Chodosh and F. Schulze. Uniqueness of asymptotically conical tangent flows. *arXiv:1901.06369*, 2019.

[DLN01] N. Dirr, S. Luckhaus, and M. Novaga. A stochastic selection principle in case of fattening for curvature flow. *Calc. Var. Partial Differential Equations*, 13(4):405–425, 2001.

[Du20] W. Du. Bounded diameter under mean curvature flow. *arXiv:2004.03769*, 2020.

[Eck04] K. Ecker. *Regularity theory for mean curvature flow.* Progress in Nonlinear Differential Equations and their Applications, 57. Birkhäuser Boston Inc., Boston, MA, 2004.

[ES91] L. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, 1991.

[GH20] P. Gianniotis and R. Haslhofer. Diameter and curvature control under mean curvature flow. *Amer. J. Math.*, 142(6):1877–1896, 2020.

[Gro99] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1999.

[Ham95] R. Hamilton. The formation of singularities in the Ricci flow. In *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, pages 7–136. Int. Press, Cambridge, MA, 1995.

[Has14] R. Haslhofer. Lectures on mean curvature flow. *arXiv:1406.7765*, 2014.

[Has15] R. Haslhofer. Uniqueness of the bowl soliton. *Geom. Topol.*, 19(4):2393–2406, 2015.

[Has16] R. Haslhofer. Lectures on curve shortening flow. *available at http://www.math.toronto.edu/roberth/*, 2016.

[Has17] R. Haslhofer. Introduction to mean curvature flow. *videos available at https://video-archive.fields.utoronto.ca/list/event/1356*, 2017.

[Hat83] A. Hatcher. A proof of the Smale conjecture, $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$. *Ann. of Math. (2)*, 117(3):553–607, 1983.

[HI01] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geom.*, 59(3):353–437, 2001.

[HK17] R. Haslhofer and B. Kleiner. Mean curvature flow of mean convex hypersurfaces. *Comm. Pure Appl. Math.*, 70(3):511–546, 2017.

[HK19] R. Haslhofer and D. Ketover. Minimal 2-spheres in 3-spheres. *Duke Math. J.*, 168(10):1929–1975, 2019.

[HN14] H.-J. Hein and A. Naber. New logarithmic Sobolev inequalities and an ϵ-regularity theorem for the Ricci flow. *Comm. Pure Appl. Math.*, 67(9):1543–1561, 2014.

[HN18] R. Haslhofer and A. Naber. Characterizations of the Ricci flow. *J. Eur. Math. Soc. (JEMS)*, 20(5):1269–1302, 2018.

[HS99a] G. Huisken and C. Sinestrari. Mean curvature flow singularities for mean convex surfaces. *Calc. Var. Partial Differential Equations*, 8(1):1–14, 1999.

[HS99b] G. Huisken and C. Sinestrari. Convexity estimates for mean curvature flow and singularities of mean convex surfaces. *Acta Math.*, 183(1):45–70, 1999.

[HS09] G. Huisken and C. Sinestrari. Mean curvature flow with surgeries of two-convex hypersurfaces. *Invent. Math.*, 175(1):137–221, 2009.

[Hui84] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.

[Hui90] G. Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.

[HW18] O. Hershkovits and B. White. Avoidance for set-theoretic solutions of mean-curvature-type flows. *arXiv:1809.03026*, 2018.

[HW20] O. Hershkovits and B. White. Nonfattening of mean curvature flow at singularities of mean convex type. *Comm. Pure Appl. Math.*, 73(3):558–580, 2020.

[Ilm93] T. Ilmanen. The level-set flow on a manifold. In *Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990)*, volume 54 of *Proc. Sympos. Pure Math.*, pages 193–204. Amer. Math. Soc., Providence, RI, 1993.

[Ilm94] T. Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.*, 108(520):x+90, 1994.

[Ilm95] T. Ilmanen. Singularities of mean curvature flow of surfaces. *notes available at https://people.math.ethz.ch/˜ilmanen/papers/sing.ps*, 1995.

[Ilm03] T. Ilmanen. Problems in mean curvature flow. *https://people.math.ethz.ch/˜ilmanen/classes/eil03/problems03.ps*, 2003.

[KL17] B. Kleiner and J. Lott. Singular Ricci flows I. *Acta Math.*, 219(1):65–134, 2017.

[KT14] K. Kasai and Y. Tonegawa. A general regularity theory for weak mean curvature flow. *Calc. Var. Partial Differential Equations*, 50(1-2):1–68, 2014.

[Ler34] J. Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.*, 63(1):193–248, 1934.

[LM20] Y. Liokumovich and D. Maximo. Waist inequality for 3-manifolds with positive scalar curvature. *arXiv:2012.12478*, 2020.

[LW18] H. Li and B. Wang. On Ilmanen's multiplicity-one conjecture for mean curvature flow with type-I mean curvature. *arXiv:1811.08654*, 2018.

[Man11] C. Mantegazza. *Lecture notes on mean curvature flow*, volume 290 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2011.

[Mul56] W. W. Mullins. Two-dimensional motion of idealized grain boundaries. *J. Appl. Phys.*, 27:900–904, 1956.

[MZ98] F. Merle and H. Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2):139–196, 1998.

[OS88] S. Osher and J. Sethian. Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations. *J. Comput. Phys.*, 79(1):12–49, 1988.

[Per02] G. Perelman. The entropy formula for the Ricci flow and its geometric applications. *arXiv:math/0211159*, 2002.

[Sch08] F. Schulze. Nonlinear evolution by mean curvature and isoperimetric inequalities. *J. Differential Geom.*, 79(2):197–241, 2008.

[Sch14] F. Schulze. Uniqueness of compact tangent flows in mean curvature flow. *J. Reine Angew. Math.*, 690:163–172, 2014.

[Sch17] F. Schulze. Introduction to mean curvature flow. *Lecture notes available at https://www.felixschulze.eu*, 2017.

[Smi82] F. Smith. On the existence of embedded minimal 2-spheres in the 3-sphere, endowed with an arbitrary Riemannian metric. *Phd thesis, Supervisor: Leon Simon, University of Melbourne*, 1982.

[Str88] M. Struwe. On the evolution of harmonic maps in higher dimensions. *J. Differential Geom.*, 28(3):485–502, 1988.

[Stu18] K.-T. Sturm. Super-Ricci flows for metric measure spaces. *J. Funct. Anal.*, 275(12):3504–3569, 2018.

[Sun18] A. Sun. Local entropy and generic multiplicity one singularities of mean curvature flow of surfaces. *arXiv:1810.08114*, 2018.

[SW09] W. Sheng and X. Wang. Singularity profile in the mean curvature flow. *Methods Appl. Anal.*, 16(2):139–155, 2009.

[SW20] F. Schulze and B. White. A local regularity theorem for mean curvature flow with triple edges. *J. Reine Angew. Math.*, 758:281–305, 2020.

[Wan11] X. Wang. Convex solutions to the mean curvature flow. *Ann. of Math. (2)*, 173(3):1185–1239, 2011.

[Wan16] L. Wang. Asymptotic structure of self-shrinkers. *arXiv:1610.04904*, 2016.

[Whi00] B. White. The size of the singular set in mean curvature flow of mean convex sets. *J. Amer. Math. Soc.*, 13(3):665–695, 2000.

[Whi02] B. White. Evolution of curves and surfaces by mean curvature. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 525–538. Higher Ed. Press, Beijing, 2002.

[Whi03] B. White. The nature of singularities in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc.*, 16(1):123–138, 2003.

[Whi05] B. White. A local regularity theorem for mean curvature flow. *Ann. of Math. (2)*, 161(3):1487–1519, 2005.

[Whi09] B. White. Currents and flat chains associated to varifolds, with an application to mean curvature flow. *Duke Math. J.*, 148(1):41–62, 2009.

[Whi15] B. White. Topics in mean curvature flow. *Lecture notes by Otis Chodosh available at http://web.stanford.edu/ ochodosh/notes.html*, 2015.

[Yip98] N. Yip. Stochastic motion by mean curvature. *Arch. Rational Mech. Anal.*, 144(4):313–355, 1998.

Department of Mathematics, University of Toronto, 40 St George Street, Toronto, ON M5S 2E4, Canada

E-mail: roberth@math.toronto.edu