Classification of noncollapsed translators in $\mathbb{R}^4$

Robert Haslhofer
(joint work with Kyeongsu Choi, Or Hershkovits)

In the analysis of mean curvature flow it is crucial to understand ancient noncollapsed flows. We recall that a mean curvature flow M_t is called ancient if it is defined for all $t \ll 0$, and noncollapsed if it is mean-convex and there is an $\alpha > 0$ such that every point $p \in M_t$ admits interior and exterior balls of radius at least $\alpha/H(p)$. In particular, thanks to the work of White [14] it is known that all blowup limits of mean-convex mean curvature flow are ancient noncollapsed flows.

In a recent breakthrough, Brendle-Choi [2, 3] and Angenent-Daskalopoulos-Sesum [1] classified all ancient noncollapsed flows in $\mathbb{R}^3$ (and similarly in $\mathbb{R}^{n+1}$ under a uniform two-convexity assumption). Specifically, they showed that any such flow is either a flat plane, a round shrinking sphere, a round shrinking cylinder, a translating bowl soliton, or an ancient oval. This in turn has been generalized in our recent proof of the mean-convex neighborhood conjecture [5, 9]. In stark contrast, the classification of ancient noncollapsed flows in higher dimensions without two-convexity assumption has remained a widely open problem.

As an important first step towards overcoming this dimension barrier, we recently classified all ancient noncollapsed flows in $\mathbb{R}^4$ assuming self-similarity:

Theorem (Choi-H.-Hershkovits [7, 8])**.** *Every noncollapsed translator in $\mathbb{R}^4$ is either $\mathbb{R}\times$2d-bowl, or the 3d round bowl, or belongs to the one-parameter family of 3d oval-bowls $\{M_k\}_{k\in(0,1/3)}$ constructed by Hoffman-Ilmanen-Martin-White* [12].

As a corollary we obtain a classification of certain blowup limits in $\mathbb{R}^4$:

Corollary (Choi-H.-Hershkovits [7, 8])**.** *For mean-convex mean curvature flow in $\mathbb{R}^4$ (or more generally in any 4-manifold), every type I blowup limit (ala Huisken) is either a round shrinking S^3, or a round shrinking $\mathbb{R}\times S^2$, or a round shrinking $\mathbb{R}^2\times S^1$, and every type II blowup limit (ala Hamilton) is either $\mathbb{R}\times$2d-bowl, or the 3d round bowl, or belongs to the one-parameter family of 3d oval-bowls.*

To sketch the main steps of the proof given a noncollapsed translator $M \subset \mathbb{R}^4$, that is neither $\mathbb{R}\times$2d-bowl nor 3d-bowl, we normalize without loss of generality such that $\mathbf{H} = e_4^\perp$. To begin with, by our no-wings theorem from [6], we have

$$\lim_{\lambda\to 0} \lambda M = \{\mu e_4 | \mu \geq 0\}. \tag{1}$$

In particular, together with a recent result of Zhu [15] this yields SO(2)-symmetry. Hence, the level sets $\Sigma^h = M \cap \{x_4 = h\}$ can be described by a renormalized profile function $v(y,\tau)$, where $\tau = -\log h$, whose analysis is governed by the one-dimensional Ornstein-Uhlenbeck operator $\mathcal{L} = \partial_y^2 - \frac{y}{2}\partial_y + 1$. Next, we show that $v(y,\tau)$ satisifies similar sharp asymptotics as the 2d ancient ovals in $\mathbb{R}^3$. We then establish a spectral uniqueness theorem, which says that if for two (suitably normalized) translators the difference of the profile functions $v_1 - v_2$ is perpendicular

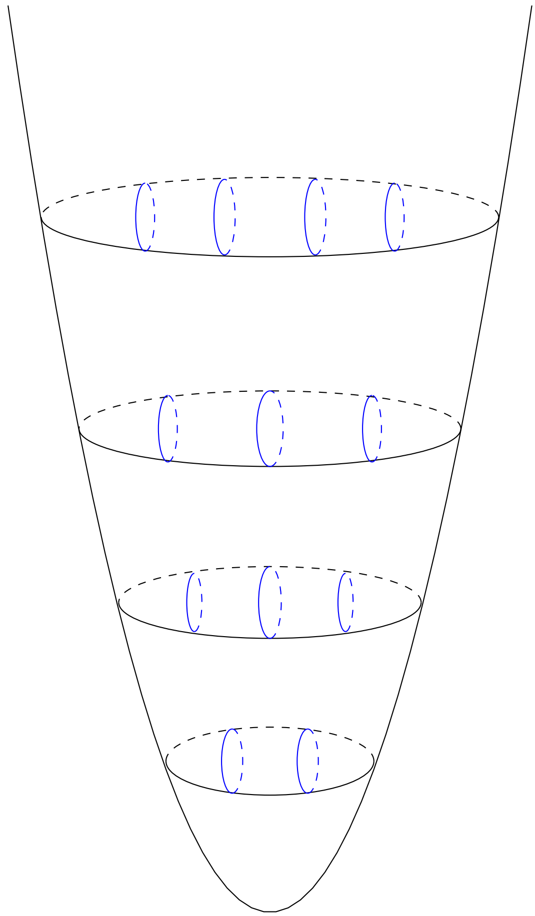

Figure 1. The oval-bowls $\{M_k\}_{k\in(0,1/3)}$ are 3-dimensional translators in $\mathbb{R}^4$, whose level sets look like 2d ovals in $\mathbb{R}^3$. They are parametrized in terms of the smallest principal curvature at the tip, and interpolate between the 3d round bowl and $\mathbb{R}\times$2d-bowl.

to the unstable and neutral eigenspace of $\mathcal{L}$, then the translators agree. We arrange this spectral condition using a delicate continuity argument. Finally, we relate the eccentricity at high levels and the tip curvature using a Rado-type argument and Lyaponov-Schmidt reduction and linearized variants of our estimates.

The result is part of a larger classification program for ancient noncollapsed flows in $\mathbb{R}^4$ that I recently introduced in joint work with Choi-Hershkovits [6] and Du [10]. In particular, in another paper with Du [11] we constructed a one-parameter family of $\mathbb{Z}_2^2 \times \mathrm{O}(2)$-symmetric ancient ovals in $\mathbb{R}^4$, which can be viewed as compact counterpart of the HIMW-family. In forthcoming work we prove:

Theorem (Choi-Daskalopoulos-Du-H.-Sesum [4])**.** *Every bubble-sheet oval for the mean curvature flow in $\mathbb{R}^4$, up to scaling and rigid motion, either is the $O(2) \times O(2)$-symmetric ancient oval from* [14]*, or belongs to the one-parameter family of $\mathbb{Z}_2^2 \times O(2)$-symmetric ancient ovals constructed in* [11]*.*

Finally, it is tempting to conjecture that similar results hold for κ-solutions in 4d Ricci flow. In particular, concerning self-similar solutions I believe:

Conjecture. *Every noncollapsed 4d steady Ricci soliton with nonnegative curvature operator is either $\mathbb{R}\times$3d-Bryant soliton, or the 4d Bryant soliton, or belongs to the one-parameter family of noncollapsed examples constructed by Lai* [13]*.*

References

[1] S. Angenent, P. Daskalopoulos, N. Sesum, *Uniqueness of two-convex closed ancient solutions to the mean curvature flow*, Ann. of Math. **192** (2020), 353–436.

[2] S. Brendle, K. Choi, *Uniqueness of convex ancient solutions to mean curvature flow in $\mathbb{R}^3$*, Invent. Math. **217** (2019), 35–76.

[3] S. Brendle, K. Choi, *Uniqueness of convex ancient solutions to mean curvature flow in higher dimensions*, Geom. Topol. **25** (2021), 2195–2234.

[4] B. Choi, P. Daskalopoulos, W. Du, R. Haslhofer, N. Sesum, *Classification of bubble-sheet ovals in $\mathbb{R}^4$*, in preparation.

[5] K. Choi, R. Haslhofer, O. Hershkovits, *Ancient low entropy flows, mean convex neighborhoods, and uniqueness*, Acta Math. (to appear)

[6] K. Choi, R. Haslhofer, O. Hershkovits, *A nonexistence result for wing-like mean curvature flows in $\mathbb{R}^4$*, preprint.

[7] K. Choi, R. Haslhofer, O. Hershkovits, *Classification of noncollapsed translators in $\mathbb{R}^4$*, preprint.

[8] K. Choi, R. Haslhofer, O. Hershkovits, *The linearized translator equation in $\mathbb{R}^4$*, in preparation.

[9] K. Choi, R. Haslhofer, O. Hershkovits, B. White, *Ancient asymptotically cylindrical flows and applications*, Invent. Math. **229** (2022), 139–241.

[10] W. Du, R. Haslhofer, *Hearing the shape of ancient noncollapsed flows in $\mathbb{R}^4$*, Comm. Pure Appl. Math. (to appear)

[11] W. Du, R. Haslhofer, *On uniqueness and nonuniqueness of ancient ovals*, preprint.

[12] D. Hoffman, T. Ilmanen, F. Martin, B. White, *Graphical translators for mean curvature flow* Calc. Var. Partial Differential Equations **58** (2019), Art. 117.

[13] Y. Lai, *A family of 3d steady gradient solitons that are flying wings*, J. Differential. Geom. (to appear)

[14] B. White, *The nature of singularities in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc. **16** (2003), 123–138.

[15] J. Zhu, SO(2)-*symmetry of translating solitons of the mean curvature flow in $\mathbb{R}^4$*, preprint.