Classification of noncollapsed translators in $\mathbb{R}^4$

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(joint work with Kyeongsu Choi, Or Hershkovits)

In the analysis of mean curvature flow it is crucial to understand ancient non-collapsed flows. We recall that a mean curvature flow $M_t$ is called ancient if it is defined for all $t \ll 0$, and noncollapsed if it is mean-convex and there is an $\alpha > 0$ such that every point $p \in M_t$ admits interior and exterior balls of radius at least $\alpha/H(p)$. In particular, thanks to the work of White [14] it is known that all blowup limits of mean-convex mean curvature flow are ancient noncollapsed flows.

In a recent breakthrough, Brendle-Choi [2, 3] and Angenent-Daskalopoulos-Sesum [1] classified all ancient noncollapsed flows in $\mathbb{R}^3$ (and similarly in $\mathbb{R}^{n+1}$ under a uniform two-convexity assumption). Specifically, they showed that any such flow is either a flat plane, a round shrinking sphere, a round shrinking cylinder, a translating bowl soliton, or an ancient oval. This in turn has been generalized in our recent proof of the mean-convex neighborhood conjecture [5, 9]. In stark contrast, the classification of ancient noncollapsed flows in higher dimensions without two-convexity assumption has remained a widely open problem.

As an important first step towards overcoming this dimension barrier, we recently classified all ancient noncollapsed flows in $\mathbb{R}^4$ assuming self-similarity:

**Theorem** (Choi-H.-Hershkovits [7, 8]). Every noncollapsed translator in $\mathbb{R}^4$ is either $\mathbb{R} \times 2d$-bowl, or the 3d round bowl, or belongs to the one-parameter family of 3d oval-bowls $\{M_k\}_{k \in (0, 1/3)}$ constructed by Hoffman-Ilmanen-Martin-White [12].

As a corollary we obtain a classification of certain blowup limits in $\mathbb{R}^4$:

**Corollary** (Choi-H.-Hershkovits [7, 8]). For mean-convex mean curvature flow in $\mathbb{R}^4$ (or more generally in any 4-manifold), every type I blowup limit (ala Huisken) is either a round shrinking $S^3$, or a round shrinking $\mathbb{R} \times S^2$, or a round shrinking $\mathbb{R}^2 \times S^1$, and every type II blowup limit (ala Hamilton) is either $\mathbb{R} \times 2d$-bowl, or the 3d round bowl, or belongs to the one-parameter family of 3d oval-bowls.

To sketch the main steps of the proof given a noncollapsed translator $M \subset \mathbb{R}^4$, that is neither $\mathbb{R} \times 2d$-bowl nor 3d-bowl, we normalize without loss of generality such that $H = e_4^\perp$. To begin with, by our no-wings theorem from [6], we have

$$\lim_{\lambda \to 0} \lambda M = \{\mu e_4 | \mu \geq 0\}.$$  

In particular, together with a recent result of Zhu [15] this yields SO(2)-symmetry. Hence, the level sets $\Sigma^h = M \cap \{x_4 = h\}$ can be described by a renormalized profile function $v(y, \tau)$, where $\tau = -\log h$, whose analysis is governed by the one-dimensional Ornstein-Uhlenbeck operator $L = \partial^2_y - \frac{\nu}{2} \partial_y + 1$. Next, we show that $v(y, \tau)$ satisfies similar sharp asymptotics as the 2d ancient ovals in $\mathbb{R}^3$. We then establish a spectral uniqueness theorem, which says that if for two (suitably normalized) translators the difference of the profile functions $v_1 - v_2$ is perpendicular
The oval-bowls $\{M_k\}_{k \in (0,1/3)}$ are 3-dimensional translators in $\mathbb{R}^4$, whose level sets look like 2d ovals in $\mathbb{R}^3$. They are parametrized in terms of the smallest principal curvature at the tip, and interpolate between the 3d round bowl and $\mathbb{R} \times 2d$-bowl.

to the unstable and neutral eigenspace of $\mathcal{L}$, then the translators agree. We arrange this spectral condition using a delicate continuity argument. Finally, we relate the eccentricity at high levels and the tip curvature using a Rado-type argument and Lyaponov-Schmidt reduction and linearized variants of our estimates.

The result is part of a larger classification program for ancient noncollapsed flows in $\mathbb{R}^4$ that I recently introduced in joint work with Choi-Hershkovits [6]
and Du [10]. In particular, in another paper with Du [11] we constructed a one-parameter family of $\mathbb{Z}_2^2 \times O(2)$-symmetric ancient ovals in $\mathbb{R}^4$, which can be viewed as compact counterpart of the HIMW-family. In forthcoming work we prove:

**Theorem** (Choi-Daskalopoulos-Du-H.-Sesum [4]). Every bubble-sheet oval for the mean curvature flow in $\mathbb{R}^4$, up to scaling and rigid motion, either is the $O(2) \times O(2)$-symmetric ancient oval from [14], or belongs to the one-parameter family of $\mathbb{Z}_2^2 \times O(2)$-symmetric ancient ovals constructed in [11].

Finally, it is tempting to conjecture that similar results hold for $\kappa$-solutions in 4d Ricci flow. In particular, concerning self-similar solutions I believe:

**Conjecture.** Every noncollapsed 4d steady Ricci soliton with nonnegative curvature operator is either $\mathbb{R} \times 3d$-Bryant soliton, or the 4d Bryant soliton, or belongs to the one-parameter family of noncollapsed examples constructed by Lai [13].

**References**