Singularities in 4d Ricci flow

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(joint work with Reto Müller)

Hamilton's Ricci flow [1], $\partial_t g = -2\mathrm{Rc}_g$, has been very successful in many cases. Highlights include Perelman's spectacular proof of the geometrization conjecture [2, 3], the Brendle-Schoen proof of the differentiable sphere theorem [4], and many deep results on the Kähler Ricci flow. However, relatively little is known in the general higher-dimensional case without strong positivity assumptions on the curvatures. In this general case extremely complicated singularities can form and a key problem is to study the nature of these singularities. The aim here is to report on our work on the compactness properties of the space of singularity models, for full details please see [5].

By Perelman's monotonicity formula the singularity models for the Ricci flow are the gradient shrinking Ricci solitons (shrinkers), given by a manifold M, a metric g and a function f such that the following equation holds:

$$\mathrm{Rc}_g + \mathrm{Hess}_g f = \tfrac{1}{2} g\,. \tag{1}$$

Solutions of (1) correspond to selfsimilar solutions of the Ricci flow, moving only by homotheties and diffeomorphisms. Shrinkers always come equipped with a natural basepoint $p \in M$, a minimum point for the potential f, in fact $f(x) \sim \frac{1}{4} d(x,p)^2$. After imposing the normalization $\int_M (4\pi)^{-n/2} e^{-f} dV = 1$, shrinkers also have a well-defined Perelman entropy,

$$\mu(g) = \int_M \big(R + |\nabla f|^2 + f - n\big)(4\pi)^{-n/2} e^{-f} dV\,. \tag{2}$$

Our first main result says that the space of shrinkers with bounded entropy and locally bounded energy is orbifold-compact in arbitrary dimensions:

Theorem 1. *For every sequence of shrinkers (M_i^n, g_i, f_i) satisfying the entropy and local energy assumptions,*

$$\mu(g_i) \geq \underline{\mu} > -\infty, \qquad \int_{B_r(p_i)} |Rm_{g_i}|^{n/2} dV_{g_i} \leq C(r) < \infty\,, \tag{3}$$

there exists a subsequence that converges to an orbifold shrinker in the pointed orbifold Cheeger-Gromov sense.

Here, the limit can have a discrete set of orbifold points modeled on finite quotients $\mathbb{R}^n/\Gamma$ ($\Gamma \subset \mathrm{O}(n)$). Away from these points the convergence is smooth. See also [6, 7, 8, 9] for related compactness results for Ricci solitons, and [10, 11, 12] for the fundamental results in the Einstein case. The strength of our Theorem 1 is that it works for noncompact manifolds and that we do not require any other assumptions, in particular no volume, diameter or pointwise curvature bounds. In fact, most interesting singularity models for the Ricci flow are noncompact, the cylinder being the most basic example. Also, assuming a lower bound for the entropy is very natural, since it is nondecreasing along the Ricci flow by Perelman's celebrated monotonicity formula [2]. In dimension four, a delicate localized Gauss-Bonnet argument even allows us to drop the assumption on energy in favor of essentially an upper bound for the Euler characteristic:

Theorem 2. *For four-dimensional shrinkers (M^4, g, f) we have the weighted L^2-estimate*

$$\int_M |Rm|^2 e^{-f} dV \leq C(\underline{\mu}, \overline{\chi}, C_{tech}) < \infty\,, \tag{4}$$

depending only on a lower bound $\underline{\mu}$ for the entropy, an upper bound $\overline{\chi}$ for the Euler characteristic, and a technical constant C_{tech} such that

$$|\nabla f|(x) \geq 1/C_{tech} \qquad \text{whenever} \qquad d(x,p) \geq C_{tech}\,. \tag{5}$$

Actually, we believe that the technical condition (5) is always satisfied. It remains an interersting open problem to prove that this is indeed the case.

Outline of the proofs. We first sketch the main steps to prove Theorem 1: Volume comparison implies the existence of a pointed Gromov-Hausdorff limit $(M_\infty, d_\infty, p_\infty)$. Using the lower bound for the entropy and the fact that the scalar curvature is locally bounded on shrinkers we obtain a lower bound for the volume of small balls (noncollapsing). The shrinker equation and the Bianchi identity yield an elliptic equation of the schematic form

$$\Delta \mathrm{Rm} = \nabla f * \nabla \mathrm{Rm} + \mathrm{Rm} + \mathrm{Rm} * \mathrm{Rm}\,. \tag{6}$$

We then prove the following ε-regularity estimate:

$$||\mathrm{Rm}||_{L^{n/2}(B_\delta(x))} \leq \varepsilon(r) \Rightarrow ||\nabla^k \mathrm{Rm}||_{L^\infty(B_{\delta/2}(x))} \leq \frac{C_k(r)}{\delta^{2+k}} ||\mathrm{Rm}||_{L^{n/2}(B_\delta(x))}\,. \tag{7}$$

A key step here is an uniform estimate for the local Sobolev constant. Putting things together we can pass to a smooth Cheeger-Gromov limit away from a discrete set of singular points. Finally, the singular points are of C^∞-orbifold type.

To get across the idea of the proof of Theorem 2, recall that the Gauss-Bonnet formula for 4-manifolds with boundary has the schematic form

$$\chi(B) = \int_B \left(|\mathrm{Rm}|^2 - |\mathrm{Rc}|^2 + R^2\right) dV + \int_{\partial B} (I\!I * \mathrm{Rm} + I\!I * I\!I * I\!I)\, dA\,. \tag{8}$$

We choose (essentially) e^{-f} as a weight function on M, use the coarea formula and apply (8). The goal is then to estimate $\int_M |\mathrm{Rm}|^2 e^{-f} dV$, by controlling all the other terms. The hardest term has the form $\int_M |\mathrm{Rc}|^3 e^{-f} dV$ and comes from the boundary term cubic in the second fundamental form (very roughly $I\!I \sim \nabla^2 f \sim \mathrm{Rc}$). At first sight, it seems impossible to control the cubic Ricci term by the bulk terms which are only quadratic. However, we have the following weighted L^3-estimate for shrinkers:

$$\int_M |\mathrm{Rc}|^3 e^{-f} dV \leq \varepsilon \int_M |\mathrm{Rm}|^2 e^{-f} dV + C(\varepsilon, \underline{\mu})\,. \tag{9}$$

Our proof of the key estimate (9) is based on a delicate use of partial integrations and soliton identities. The proof of Theorem 2 can then be finished by estimating and absorbing all the remaining terms.

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