Ricci curvature and martingales

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(joint work with Aaron Naber)

The main goal of this talk, based on [1], is to explain how bounded Ricci curvature can be understood by analyzing the evolution of martingales on path space, generalizing the well known and important principles of how lower bounds on Ricci curvature can be understood by analyzing the heat flow.

To put things into context, let us recall that the starting point for most of the analysis on spaces with Ricci curvature bounded below, say by a constant $-\kappa$, is the classical Bochner inequality

$$\tfrac{1}{2}\Delta|\nabla u|^2 \geq \langle \nabla\Delta u, \nabla u\rangle + |\nabla^2 u|^2 - \kappa|\nabla u|^2\,. \tag{1}$$

Using the Bochner inequality it is a simply exercise to show that Ricci bounded below by $-\kappa$ is equivalent to several other geometric-analytic estimates, e.g. the following sharp gradient estimate for the heat flow

$$|\nabla H_t u| \leq e^{\frac{\kappa}{2}t} H_t|\nabla u|\,. \tag{2}$$

In contrast to the well developed theory of Ricci curvature bounded below, until recently there was no characterization available at all for spaces with bounded Ricci curvature. This characterization problem has been solved recently by Naber [3]. The key insight was that to understand two-sided bounds for Ricci curvature, and not just lower bounds, one should do analysis on path space PM, instead of analysis on M. By definition, given a complete Riemannian manifold M, its path space $PM = C([0,\infty), M)$ is the space of continuous curves in M. Path space comes equipped with a family of natural probability measures, the Wiener measure Γ_x of Brownian motion starting at $x \in M$. Path space also comes equipped with a natural one parameter family of gradients, the t-parallel gradients $\nabla_t^{\|}$ $(t \geq 0)$. Using this framework, it was proved in [3] that the Ricci curvature of M is bounded by a constant κ if and only if the sharp gradient estimate

$$\left|\nabla_x \int_{PM} F\, d\Gamma_x\right| \leq \int_{PM} \left(|\nabla_0^{\|}F| + \int_0^\infty \frac{\kappa}{2} e^{\kappa t/2}|\nabla_t^{\|}F|\, dt\right) d\Gamma_x \tag{3}$$

holds for all test functions $F : PM \to \mathbb{R}$. In the simplest case of one-point test functions, i.e. functions of the form $F(\gamma) = u(\gamma(t))$ where $u : M \to \mathbb{R}$ and t is fixed, the infinite dimensional gradient estimate (3) reduces to the finite dimensional gradient estimate (2). The gradient estimate (3) can be used to define a weak notion of Ricci curvature for metric measure spaces.

While [3] gives a way to generalize certain estimates for lower Ricci curvature on M to estimates for bounded Ricci curvature on PM, e.g. the finite dimensional gradient estimate (2) to the infinite dimensional gradient estimate (3), what hasn't been answered yet is the following question:

Is there any way to generalize the Bochner inequality (1) *from M to PM?*

This question has been the guiding principle for the present work. Given that the Bochner formula is the starting point for most of the theory of lower Ricci, such a generalization is clearly valuable for the theory of bounded Ricci curvature.

The first main point we wish to explain is that martingales on PM are the correct generalization of the heat flow on M. Recall that a *martingale* on P_xM is a Σ_t-adapted integrable stochastic process $F_t : P_xM \to \mathbb{R}$ such that

$$F_{t_1} = E_x[F_{t_2} \,|\, \Sigma_{t_1}] \qquad (t_1 \leq t_2). \tag{4}$$

Here, the right hand side denotes the conditional expectation value on P_xM given the σ-algebra Σ_{t_1}, of events which are observable until time t_1. The simplest examples of martingales on path space have the form

$$F_t(\gamma) = \begin{cases} H_{T-t}u(\gamma(t)), & \text{if } t < T \\ u(\gamma(T)), & \text{if } t \geq T, \end{cases} \tag{5}$$

where $u : M \to \mathbb{R}$ and T are fixed, and thus are indeed given by the (backwards) heat flow on M.

We found that the correct generalization of the Bochner formula (1) on M is given by a certain evolution equation for martingales on PM. To get there, we start with by reformulating the martingale representation theorem and the Clark-Ocone formula in the form

$$dF_t = \langle \nabla^{||}_t F_t, dW_t \rangle \,. \tag{6}$$

Expressed this way, we can view the martingale equation as an evolution equation on path space. We then proceed by computing various evolution equations for associated quantities on path space. In particular, if $F_t : P_xM \to \mathbb{R}$ is a martingale on path space, and $s \in \mathbb{R}$ is fixed, then its s-parallel gradient $\nabla^{||}_s F_t : P_xM \to T_xM$ satisfies the stochastic equation

$$d\nabla^{||}_s F_t = \langle \nabla^{||}_t \nabla^{||}_s F_t, dW_t \rangle + \frac{1}{2}\mathrm{Ric}_t(\nabla^{||}_t F_t)\, dt + \nabla^{||}_s F_s \,\delta_s(t) dt \,, \tag{7}$$

where $\langle \mathrm{Ric}_t(X), Y \rangle = \mathrm{Ric}(P_t^{-1}X, P_t^{-1}Y)$ and $P_t = P_t(\gamma) : T_{\gamma(t)}M \to T_xM$ is stochastic parallel transport. Combining this with the Ito formula we obtain

$$\begin{aligned} d|\nabla^{||}_s F_t|^2 = {} & \langle \nabla^{||}_t |\nabla^{||}_s F_t|^2, dW_t \rangle \\ & + |\nabla^{||}_t \nabla^{||}_s F_t|^2 dt + \mathrm{Ric}_t\big(\nabla^{||}_t F_t, \nabla^{||}_s F_t\big)\, dt + |\nabla^{||}_s F_s|^2 \delta_s(t) dt, \end{aligned} \tag{8}$$

which is the correct generalization of the Bochner formula to path space.

We will now discuss four applications of our calculus on path space. First, it yields a shorter proof of the characterizations from [3]. For illustration, if $\mathrm{Ric} = 0$ then by (7) the process $t \mapsto |\nabla^{||}_s F_t|$ is a submartingale. Thus, by the very definition of a submartingale we get

$$|\nabla^{||}_s F_t| \leq E_x\left[|\nabla^{||}_s F_T| \,\big|\, \Sigma_t \right] \qquad (t \leq T). \tag{9}$$

Taking the limit $T \to \infty$, and specializing to $s = t = 0$, this implies the $\kappa = 0$ case of the infinite dimensional gradient estimate (3):

$$\left| \nabla_x \int_{PM} F \, d\Gamma_x \right| \leq \int_{PM} |\nabla_0^{||} F| \, d\Gamma_x. \tag{10}$$

Other characterizations, and estimates for $\kappa \neq 0$, can be proven with similar ease.

Second, the gradient estimate (3) can be strengthened to the family of estimates

$$|\nabla_s^{||} F_t| \leq E\left[|\nabla_s^{||} F| + \frac{\kappa}{2} \int_t^\infty e^{\frac{\kappa}{2}(r-t)} |\nabla_r^{||} F| \, dr \,\middle|\, \Sigma_t \right]. \tag{11}$$

Third, we obtain new characterizations of bounded Ricci curvature. In particular, $|\mathrm{Ric}| \leq \kappa$ is equivalent to the full Bochner inequality on path space

$$\begin{aligned} d|\nabla_s^{||} F_t|^2 \geq \langle \nabla_t^{||} |\nabla_s^{||} F_t|^2, dW_t \rangle \\ + |\nabla_t^{||} \nabla_s^{||} F_t|^2 dt - \kappa |\nabla_t^{||} F_t| |\nabla_s^{||} F_t| \, dt + |\nabla_s^{||} F_s|^2 \delta_s(t) dt, \end{aligned} \tag{12}$$

as well as the weak Bochner inequality on path space

$$d|\nabla_s^{||} F_t| \geq \langle \nabla_t^{||} |\nabla_s^{||} F_t|, dW_t \rangle \; - \tfrac{\kappa}{2} |\nabla_t^{||} F_t| \, dt. \tag{13}$$

Forth, we obtain new Hessian estimates for martingales on the path space of manifolds with bounded Ricci curvature, e.g.

$$\begin{aligned} \int_{PM} |\nabla_s^{||} F_s|^2 \, d\Gamma_x + \int_0^T \int_{PM} |\nabla_t^{||} \nabla_s^{||} F_t|^2 \, d\Gamma_x \, dt \\ \leq e^{\frac{\kappa}{2}(T-s)} \int_{PM} \left(|\nabla_s^{||} F|^2 + \frac{\kappa}{2} \int_s^T e^{\frac{\kappa}{2}(t-s)} |\nabla_t^{||} F|^2 \, dt \right) d\Gamma_x. \end{aligned} \tag{14}$$

Combined with Doob's inequality this generalizes the classical $L^\infty H^1 \cap L^2 H^2$ estimate for the heat flow on M.

The methods can also be adapted to the time-dependent setting, and thus also provide a useful tool for the study of Ricci flow in the framework of [2].

References

[1] R. Haslhofer, A. Naber, *Ricci curvature and martingales*, in preparation.
[2] R. Haslhofer, A. Naber, *Characterizations of the Ricci flow*, JEMS (to appear).
[3] A. Naber, *Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces.*