Quantitative stratification and the regularity of mean curvature flow

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(joint work with Jeff Cheeger and Aaron Naber)

The aim here is to report on our recent estimates and quantitative regularity results for the mean curvature flow of n-dimensional surfaces in $\mathbb{R}^N$. For full details please see [2], for related results for the harmonic map flow please see [3].

Recall first that smooth solutions of the mean curvature flow are given by a smooth family of submanifolds $M_t^n \subset \mathbb{R}^N$ satisfying the evolution equation,

$$\partial_t x = H(x), \qquad x \in M_t\,. \tag{1}$$

More generally though, M_t is a family of Radon-measures that is integer n-rectifiable for almost all times and satisfies (1) in the weak sense of Brakke, i.e.

$$\overline{D}_t \int \varphi dM_t \leq \int \left(-\varphi H^2 + \nabla\varphi \cdot H\right) dM_t \tag{2}$$

for all nonnegative test functions φ, where $\overline{D}_t$ is the limsup of difference quotients. Brakke flows enjoy wonderful existence and compactness properties, see the fundamental work of Brakke and Ilmanen [1, 8]. The main problem is then to investigate their regularity.

Our results build upon the deep regularity theory of Brian White [9, 10, 11, 12], which we briefly recall now: Given a Brakke flow $\mathcal{M} = \{(M_t, t)\}$ he considered the stratification of the singular set $\mathcal{S} \subset \mathcal{M}$,

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^{n+1} \subset \mathcal{S}\,, \tag{3}$$

where by definition $X = (x,t) \in \mathcal{S}^j$ if and only if no tangent flow at X has more than j symmetries. For general Brakke flows White first proved the (parabolic) Hausdorff dimension estimate

$$\dim \mathcal{S}^j \leq j\,. \tag{4}$$

For the flow of mean-convex hypersurfaces, he then proved the deep result

$$\mathcal{S} = \mathcal{S}^{n-1}\,, \tag{5}$$

and thus that the singular set has (parabolic) Hausdorff dimension at most $n-1$. This is based on many clever arguments, ruling out in particular higher multiplicities. He also gives a precise description of the singularities in this mean-convex case: all tangent flows are spheres, cylinders or planes of multiplicity one.

Let us now come to the general idea of quantitative stratification: Recall first that the standard stratification / dimension reduction method, a method to prove Hausdorff dimension estimates like (4), was introduced first in the context of geometric measure theory and later applied successfully in all kind of situations in geometric analysis. In the quantitative stratification, introduced recently by Cheeger-Naber in the elliptic setting [4, 5] and developed and applied now in the parabolic setting by Cheeger-Haslhofer-Naber [2, 3], we replace the singular strata $\mathcal{S}^j$ by quantitative singular strata $\mathcal{S}^j_{\eta,r}$ ($\eta > 0$, $0 < r < 1$). We then show that tubular neighborhoods of $\mathcal{S}^j_{\eta,r}$ have small volume, and that away from a bad set of small volume we get definite estimates on balls of definite size.

Concretely, let $\mathcal{M}_{X,s}$ be the flow obtained by shifting X to the origin and rescaling parabolically by $1/s$, and let d be a suitable distance function on the space of Brakke flows on the unit ball. Then

$$\mathcal{S}^j_{\eta,r} := \{X \in \mathcal{M} : d(\mathcal{M}_{X,s}, \mathcal{N}) > \eta \text{ for all } r \leq s \leq 1 \\ \text{and all selfsimilar } \mathcal{N} \text{ with more than } j \text{ symmetries}\}\,. \tag{6}$$

Theorem 1. *For all $\varepsilon, \eta > 0$, $\Lambda < \infty$ and N, there exists $C = C(\varepsilon, \eta, \Lambda, N) < \infty$ such that: If $\mathcal{M}$ is a Brakke flow, defined on a space-time ball $B_2 \subset \mathbb{R}^{N,1}$ and with mass at most Λ, then its j-th quantitative singular stratum satisfies*

$$\mathrm{Vol}\left(T_r(\mathcal{S}^j_{\eta,r}) \cap B_1\right) \leq C r^{N+2-j-\varepsilon} \qquad (0 < r < 1)\,. \tag{7}$$

By virtue of $\cup_{\eta>0} \cap_{r>0} \mathcal{S}^j_{\eta,r} = \mathcal{S}^j$, we recover the standard Hausdorff dimension estimate (4), but of course our theorem contains much more quantitative information about the singular set than just its dimension.

Coming to applications, we focus on Brakke flows starting at hypersurfaces $M_0 \subset \mathbb{R}^{n+1}$ (smooth, compact, embedded) that are k-convex, i.e. $\lambda_1 + \ldots \lambda_k \geq 0$ where $\lambda_1 \leq \ldots \leq \lambda_n$ denote the principal curvatures. Special instances are the convex case ($k = 1$) with Huisken's classical result, the 2-convex case where Huisken-Sinestrari constructed a mean curvature flow with surgery, and the general mean-convex case ($k = n$) with White's regularity theory. Building on the work of White via elliptic regularization we prove:

Theorem 2. *Let $\mathcal{M}$ be a Brakke flow starting at a k-convex hypersurface. Then any selfsimilar limit flow $\mathcal{N} = \lim \mathcal{M}_{X_\alpha, r_\alpha}$ with at least k symmetries is in fact a static multiplicity one plane. In particular, for every singular point $X \in \mathcal{S}$ all tangent flows are shrinking spheres or cylinders*

$$\mathbb{R}^j \times S^{n-j} \qquad \text{with} \qquad 0 \leq j < k\,. \tag{8}$$

The idea is now to combine Theorem 1 and Theorem 2, to obtain our main regularity result for k-convex mean curvature flows. To state it, for $X = (x,t) \in \mathcal{M}$ we define the regularity scale $r_{\mathcal{M}}(X)$ as the supremum of $0 \leq r \leq 1$ such that $M_{t'} \cap B_r(x)$ is a smooth graph for all $t - r^2 < t' < t + r^2$ and such that

$$\sup_{X' \in \mathcal{M} \cap B_r(X)} r\,|A|(X') \leq 1\,, \tag{9}$$

where A is the second fundamental form. For $0 < r < 1$ we then define the r-bad set

$$\mathcal{B}_r := \{X = (x,t) \in \mathcal{M} \,|\, r_{\mathcal{M}}(X) < r\}\,. \tag{10}$$

Theorem 3. *Let $\mathcal{M}$ be a Brakke flow starting at a k-convex hypersurface $M_0 \subset \mathbb{R}^{n+1}$ and $\varepsilon > 0$. Then there exists a constant $C = C(M_0, \varepsilon) < \infty$ such that we have the volume estimate*

$$\mathrm{Vol}(T_r(\mathcal{B}_r)) \leq C r^{n+4-k-\varepsilon} \qquad (0 < r < 1)\,, \tag{11}$$

for the r-tubular neighborhood of the bad set $\mathcal{B}_r$. In particular, the (parabolic) Minkowski dimension of the singular set is at most $k-1$.

As a consequence, we obtain L^p-estimates for the inverse regularity scale, and thus in particular L^p-estimates for the second fundamental form and its derivatives.

Corollary 4. *Let $\mathcal{M}$ be a Brakke flow starting at a k-convex hypersurface $M_0 \subset \mathbb{R}^{n+1}$. Then for every $0 < p < n+1-k$ there exists a constant $C = C(M_0, p) < \infty$ such that*

$$\int r_{\mathcal{M}}^{-p} dM_t \leq C \qquad \text{and} \qquad \int_0^\infty \int r_{\mathcal{M}}^{-(p+2)} dM_t \, dt \leq C\,. \tag{12}$$

In particular, we have L^p-estimates for the second fundamental form,

$$\int |A|^p dM_t \leq C \qquad \text{and} \qquad \int_0^\infty \int |A|^{p+2} dM_t \, dt \leq C\,, \tag{13}$$

and also L^p-estimates for the derivatives of the second fundamental form,

$$\int |\nabla^\ell A|^{\frac{p}{\ell+1}} dM_t \leq C_\ell \qquad \text{and} \qquad \int_0^\infty \int |\nabla^\ell A|^{\frac{p+2}{\ell+1}} dM_t \, dt \leq C_\ell\,, \tag{14}$$

for some constants $C_\ell = C_\ell(M_0, p) < \infty$ ($\ell = 1, 2, \ldots$).

Using very different techniques, some special cases of this corollary ($k = 2$, first singular time) have been obtained previously by Head [7] and Ecker [6].

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