The moduli space of 2-convex embedded spheres

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To put things into context, let us start with a general discussion of the moduli space of embedded n-spheres in $\mathbb{R}^{n+1}$, i.e. the space

$$\mathcal{M}_n = \mathrm{Emb}(S^n, \mathbb{R}^{n+1})/\mathrm{Diff}(S^n).$$

In 1959, Smale proved that the space of embedded circles in the plane is contractible [3], i.e.

$$\mathcal{M}_1 \simeq *.$$

In particular, the assertion $\pi_0(\mathcal{M}_1) = 0$ is equivalent to the smooth version of the Jordan-Schoenflies theorem, and the assertion $\pi_1(\mathcal{M}_1) = 0$ is equivalent to Munkres' theorem that $\mathrm{Diff}_+(S^2)$ is path-connected [4].

Moving to $n = 2$, Smale conjectured that the space of embedded 2-spheres in $\mathbb{R}^3$ is also contractible, i.e. that

$$\mathcal{M}_2 \simeq *.$$

In 1983, Hatcher proved the Smale conjecture [5]. The assertion $\pi_0(\mathcal{M}_2) = 0$ is equivalent to Alexander's strong form of the three dimensional Schoenflies theorem [6]. The assertion $\pi_1(\mathcal{M}_2) = 0$ is equivalent to Cerf's theorem that $\mathrm{Diff}_+(S^3)$ is path-connected [7], which had wide implications in differential topology.

For $n \geq 3$ not a single homotopy group of $\mathcal{M}_n$ is known. Most importantly:

The naive guess that $\mathcal{M}_n \simeq *$ for all n is completely false.

Indeed, if $\mathcal{M}_n$ were contractible for every n, then arguing as in [5] we could infer that $\mathcal{D}_n := \mathrm{Diff}(D^{n+1} \,\mathrm{rel}\, \partial D^{n+1}) \simeq *$ for every n. However, it is known that $\mathcal{D}_n$ has non-vanishing homotopy groups for every $n \geq 4$. Even more strikingly, for every $n \geq 6$ there are infinitely many i such that $\pi_i(\mathcal{D}_n) \neq 0$ [8].

In the view of the topological complexity of $\mathcal{M}_n$ for general n, it is an interesting question whether one can still derive some positive results on the space of embedded n-spheres under some curvature conditions. Such results would show that the all the non-trivial topology of $\mathcal{M}_n$ is caused by embeddings of S^n that are geometrically very far away from the canonical one. Motivated by the topological classification result from [9], we consider 2-convex embeddings, i.e. embeddings such that the sum of the smallest two principle curvatures is positive. Clearly, 2-convexity is preserved under reparametrizations. We can thus consider the subspace

$$\mathcal{M}_n^{\text{2-conv}} \subset \mathcal{M}_n$$

of 2-convex embedded n-spheres in $\mathbb{R}^{n+1}$. We propose the following higher dimensional Smale type conjecture.

Conjecture. *The moduli space of 2-convex embedded n-spheres in $\mathbb{R}^{n+1}$ is contractible, for every dimension n, i.e.*

$$\mathcal{M}_n^{\textit{2-conv}} \simeq *.$$

We recently confirmed the π_0-part of the conjecture:

Theorem (Buzano-Haslhofer-Hershkovits [1])**.** *The moduli space of 2-convex embedded n-spheres in $\mathbb{R}^{n+1}$ is path-connected, for every dimension n, i.e.*

$$\pi_0(\mathcal{M}_n^{\textit{2-conv}}) = 0.$$

To the best of our knowledge, our theorem is the first topological result about a moduli space of embedded spheres for any $n \geq 3$ (except of course for the moduli space of convex embedded spheres, which is easily seen to be contractible).

Our proof uses mean curvature flow with surgery. Surgery for 2-convex mean curvature flow has been implemented first by Huisken-Sinestrari [9], and more recently by Haslhofer-Kleiner [10] and Brendle-Huisken [2]. We use the approach from [10]. Besides being comparably short, this approach has the advantage that it works in every dimension and that it comes with the canonical neighborhood theorem [10, Thm. 1.22], which is quite crucial for our topological application.

Given a two 2-convex embedded sphere $M_0 \subset \mathbb{R}^{n+1}$, we consider its mean curvature flow with surgery $\{M_t\}_{t\in[0,\infty)}$ as provided by the existence theorem from [10, Thm. 1.21]. There are finitely many times where suitable necks are replaced by standard caps and/or where connected components with specific geometry and topology are discarded. The flow always becomes extinct in finite time $T < \infty$.

We first analyze the discarded components. By the canonical neighborhood theorem [10, Thm. 1.22] and the topological assumption on M_0, each connected component which gets discarded is either a convex sphere of controlled geometry or a capped off chain of ε-necks. This information is enough to construct an explicit path in $\mathcal{M}_n^{\text{2-conv}}$ connecting any discarded component to a round sphere.

We then prove by backwards induction on the surgery times that at each time every connected component is isotopic via 2-convex embeddings to what we call a marble tree. Roughly speaking, a marble tree is a connected sum of spheres (marbles) along admissible curves (strings), that does not contain any loop.

At the extinction time, by the above discussion, every connected component is isotopic via 2-convex embeddings to a round sphere, i.e. a marble tree with just a single marble and no strings. The key for the induction step is to glue together the isotopies of the pieces. To this end, we prove the existence of a connected sum operation that preserves 2-convexity and embeddedness, and that is continuous for families of suitable gluing configurations. Roughly speaking, the key for the gluing is to choose the string radius r_s much smaller than the trigger scale H_{trig}^{-1} associated to the flow with surgery, so that the different scales barely interact.

Finally, we prove by induction on the number of marbles that every marble tree is isotopic via 2-convex embeddings to a round sphere.

References

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