Minimal two-spheres in three-spheres

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(joint work with Dan Ketover)

The min-max method goes back to Birkhoff, who in 1917 proved:

Theorem 1 (Birkhoff [1]). *Any closed Riemannian two-sphere contains at least one closed geodesic.*

Loosely speaking, Birkhoff considered sweepouts of the two-sphere by closed curves, and argued that the longest slice in a sweepout that is pulled tight is a closed geodesic. There are also higher non-trivial families of curves one can consider to produce more geodesics:

Theorem 2 (Lusternik-Schnirelmann [9, 4]). *Any closed Riemannian two-sphere contains at least three simple closed geodesics.*

In one higher dimension, one can consider sweepouts of three-spheres by two-spheres, and hope to produce an embedded minimal two-sphere. In 1983, Simon and Smith carried this out (adapting the more general min-max theory of Almgren and Pitts to the case of surfaces with fixed topology) and proved:

Theorem 3 (Simon-Smith [10]). *Let M be a three-manifold diffeomorphic to $\mathbb{S}^3$. Then M contains an embedded minimal two-sphere.*

In analogy with the case of simple closed geodesics on two-spheres, there are also higher parameter families of two-spheres on three-spheres that one can consider. One might hope that the families detecting the relevant cohomology classes $\alpha, \ldots, \alpha^4$ produce via min-max four distinct minimal two-spheres. The major difficulty is the phenomenon of *multiplicity*. Namely, it could happen that the min-max spheres associated with the second, third and fourth family, just give the sphere associated to the first family counted with higher integer multiplicities.

Using combined efforts from min-max theory and mean curvature flow we prove:

Theorem 4 (Haslhofer-Ketover [5]). *Let M be a three-manifold diffeomorphic to $\mathbb{S}^3$ and endowed with a bumpy metric. Then M contains at least 2 embedded minimal two-spheres. More precisely, exactly one of the following alternatives holds:*

(1) *M contains at least 1 stable embedded minimal two-sphere, and at least 2 embedded minimal two-spheres of index one.*
(2) *M contains no stable embedded minimal two-sphere, at least 1 embedded minimal two-sphere Γ_1 of index one, and at least 1 embedded minimal two-sphere Γ_2 of index two. In this case, $|\Gamma_2| < 2|\Gamma_1|$.*

We note that White [11] previously proved the existence of at least 2 minimal two-spheres in the special case that M has positive Ricci curvature.

A natural family of examples to illustrate Theorem 4 are ellipsoids. Namely, given $a > b > c > d > 0$, consider the ellipsoid

$$E(a,b,c,d) := \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\} \subset \mathbb{R}^4.$$

Observe that E contains at least 4 minimal 'planar' two-spheres, which are given by the intersection with the coordinates hyperplanes $\{x_i = 0\}$. However, by the area estimate $|\Gamma_2| < 2|\Gamma_1|$, if $a \gg b$ the second minimal two-sphere $\Gamma_2(a) \subset E$ produced by Theorem 4 is not planar. Moreover, as $a \to \infty$, the minimal two-spheres $\Gamma_2(a)$ converge as varifolds to a minimal two-sphere with multiplicity two.

Let us now sketch the main ideas of the proof of Theorem 4.

If M admits a stable embedded minimal two-sphere, then the manifold is a kind of dumbbell. Considering 1-parameter sweep-outs of both halves and using [7] we show that each half contains an unstable two-sphere of index one in its interior.

Let us now consider the case that M does not contain any stable embedded minimal two-spheres. Using Simon-Smith's existence theorem (Theorem 3) we obtain 1 embedded minimal two-sphere Γ_1 of index one. Sliding the Simon-Smith sphere a bit to both sides we can decompose $M = D_1 \cup Z \cup D_2$ where Z is the short cylindrical region obtained by sliding the Simon-Smith sphere around, and D_1 and D_2 are smooth embedded 3-discs with mean convex boundary. To proceed, we prove the following general theorem establishing the existence of smooth mean convex foliations in three-manifolds:

Theorem 5 (Haslhofer-Ketover [5])**.** *Let $D \subset M^3$ be a smooth three-disc with mean convex boundary. Then exactly one of the following alternatives holds true:*

(1) *There exists an embedded stable minimal two-sphere $\Gamma \subset Int(D)$.*
(2) *There exists a smooth foliation $\{\Sigma_t\}_{t\in[0,1]}$ of D by mean convex embedded two-spheres.*

Let us first explain how to finish the proof of Theorem 4 using Theorem 5.

Recalling that $M = D_1 \cup Z \cup D_2$ and using the foliations of D_1 and D_2 produced by Theorem 5 we can build an optimal foliation of M, by which we mean a smooth foliation $\{\Sigma_t\}_{t\in[-1,1]}$ of M by two-spheres so that the Simon-Smith sphere sits in the middle of the foliation as Σ_0 and all other slices have less area. From the one parameter family $\{\Sigma_t\}$ we can then form a two parameter family $\{\Sigma_{s,t}\}$ detecting α^2 and such that

$$\sup_{s,t} |\Sigma_{s,t}| < 2|\Gamma_1|. \tag{1}$$

Roughly speaking $\Sigma_{s,t}$ looks like Σ_s connected to Σ_t along a small neck, which we open up near $(s,t) \approx (0,0)$, using the catenoid estimate from [8].

The area bound (1) ensures that min-max for our two-parameter family doesn't simply produce Γ_1 with multiplicity two. We conclude that there exists an embedded minimal two-sphere Γ_2 with $|\Gamma_1| < |\Gamma_2| < 2|\Gamma_1|$ and index two.

To obtain some intuition for Theorem 5 (which is of independent interest), imagine that the disc D evolves by mean curvature flow. Recall that mean-convexity is preserved under mean curvature flow. In the simplest possible scenario, the mean curvature flow of D remains smooth and either becomes extinct in finite time in a round point, giving the foliation from (2), or converges for $t \to \infty$ to a minimal embedded two-sphere, giving (1). Of course, in general the situation is more complicated since the mean curvature flow typically develops local singularities. One way to continue the flow through singularities is given by the level set method, and in fact our proof shows that case (2) happens if and only if the level set flow becomes extinct in finite time. The main issue however is that the foliation produced by the level set flow is in general singular.

To produce a smooth foliation instead of a singular foliation we use mean curvature flow with surgery. Mean curvature flow with surgery in general ambient manifolds has been constructed first by Brendle-Huisken [2]. However, since we also need a canonical neighborhood theorem for our application we instead extend the approach from Haslhofer-Kleiner [6] to the setting of general ambient manifolds. We then combine the existence theorem, the canonical neighborhood theorem, and methods from the recent topological application of mean curvature flow with surgery by Buzano-Haslhofer-Hershkovits [3] to produce the desired smooth foliation.

References

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