Differential Harnack inequalities on path space

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(joint work with Eva Kopfer and Aaron Naber)

Consider a Riemannian manifold (M^n, g) and for $f : M \to \mathbb{R}$ denote by $f_t : M \to \mathbb{R}$ the solution of the heat equation $(\partial_t - \Delta) f_t = 0$ with $f_0 = f$. The classical Li-Yau differential Harnack inequality [3] tells us that if f is nonnegative and $\mathrm{Rc} \geq 0$, then we have

$$\frac{\Delta f_t}{f_t} - \frac{|\nabla f_t|^2}{f_t^2} + \frac{n}{2t} \geq 0\,. \tag{1}$$

Hamilton [1], under the more restrictive assumption that $\sec \geq 0$ and $\nabla \mathrm{Rc} = 0$, proved the Hessian version of (1) given by

$$\frac{\nabla^2 f_t}{f_t} - \frac{\nabla f_t \otimes \nabla f_t}{f_t^2} + \frac{g}{2t} \geq 0\,. \tag{2}$$

In [2], we found differential Harnack inequalities on path space, which can be viewed as generalizations of the above classical inequalities on manifolds.

Recall that path space $P_x M$ is the space of all continuous curves $\gamma : [0, \infty) \to M$ starting at x. It comes equipped with the Wiener measure $\mathbb{P}_x$ of Brownian motion, which is characterized by the formula

$$\mathbb{P}_x[\gamma_{t_1} \in U_1, \ldots, \gamma_{t_k} \in U_k] = \int_{U_1 \times \ldots \times U_k} \rho_{t_1}(x, dy_1) \ldots \rho_{t_k - t_{k-1}}(y_{k-1}, dy_k), \tag{3}$$

where $\rho_t(x, dy)$ denotes the heat kernel measure.

We consider the following new notions of gradients, Hessians and Laplacians on path space. Let $\varphi : [0, \infty) \to \mathbb{R}$ be an H^1_0-function. The φ-gradient $\nabla_\varphi F : P_x M \to T_x M$ is defined by

$$\langle \nabla_\varphi F, v \rangle = \frac{d}{ds}\Big|_{s=0} F(\gamma_s)\,, \tag{4}$$

where γ_s is a family of curves with $\partial_s|_{s=0}\gamma_s = \varphi V$. Here, V denotes the vector field along γ obtained by parallel translating v. Similarly, the φ-Hessian $\mathrm{Hess}_\varphi F : P_x M \to T_x^* M \otimes T_x^* M$ is defined by

$$\mathrm{Hess}_\varphi F(v, v) = \frac{d^2}{ds^2}\Big|_{s=0} F(\gamma_s)\,, \tag{5}$$

where γ_s is a family of curves with $\partial_s|_{s=0}\gamma_s = \varphi V$ and $\nabla_{\varphi V}\big(\partial_s \gamma_s\big) = 0$. Finally, the φ-Laplacian $\Delta_\varphi F : P_x M \to \mathbb{R}$ obtained by tracing the φ-Hessian:

$$\Delta_\varphi F = \mathrm{tr}\big(\mathrm{Hess}_\varphi F\big). \tag{6}$$

Let us now state our main theorem in the context of Ricci flat spaces:

Theorem. *Let M be a Ricci-flat manifold, and let $F : P_xM \to \mathbb{R}^+$ be a nonnegative function on path space. Then, for all $\varphi \in H^1_0(\mathbb{R}^+)$ we have the inequality*

$$\frac{\mathbb{E}_x\left[\Delta_\varphi F\right]}{\mathbb{E}_x[F]} - \frac{\left|\mathbb{E}_x\left[\nabla_\varphi F\right]\right|^2}{\mathbb{E}_x[F]^2} + \frac{n}{2}||\varphi||^2 \geq 0. \tag{7}$$

For illustration, consider a function $F : P_0M \to \mathbb{R}^+$ which only depends on the value of the curve at a single time. Namely, let $F(\gamma) = f(\gamma_t)$, where $f : M \to \mathbb{R}^+$ and $t > 0$ are fixed. Let $\varphi(s) = \frac{s}{t}$ for $s \leq t$ and $\varphi(s) = 1$ for $s \geq t$. First, note that $||\varphi||^2 = \frac{1}{t}$. Next, it is an instructive exercise to compute

$$\begin{aligned} \nabla_\varphi F(\gamma) &= P_t(\gamma)\nabla f(\gamma_t)\,, \\ \Delta_\varphi F(\gamma) &= \Delta f(\gamma_t)\,, \end{aligned} \tag{8}$$

where $P_t(\gamma) : T_{\gamma(t)}M \to T_xM$ denotes parallel transport. Finally, using this and the Feynman-Kac formula we can derive the equalities

$$\begin{aligned} \mathbb{E}_x[F] &= \int_M f(y)\rho_t(x,dy) = f_t(x)\,, \\ \mathbb{E}_x[\Delta_\varphi F] &= \Delta f_t(x)\,, \\ \mathbb{E}_x[\nabla_\varphi F] &= \nabla f_t(x)\,. \end{aligned} \tag{9}$$

Plugging all of this into (7) we obtain precisely the Li-Yau Harnack inequality (1).

Plugging in a (smeared) delta function, our main theorem implies

$$-\Delta_\varphi \ln \mathbb{P}_x \leq \frac{n}{2} \tag{10}$$

for all normalized φ, which can be viewed as Laplace comparison theorem for the Wiener measure on the path space of Ricci-flat manifolds.

Finally, we also have a differential Matrix Harnack inequality on path space of general manifolds, meant to generalize Hamilton's Matrix Harnack (2):

Theorem. *Let $F : P_xM \to \mathbb{R}^+$ be a nonnegative Σ_T-measurable function on path space. Then, for every $\varphi \in H^1_0(\mathbb{R}^+)$ we have the inequality*

$$\begin{aligned} &\frac{\mathbb{E}_x[\mathrm{Hess}_{\vdash,\varphi}F]}{\mathbb{E}_x[F]} - \frac{\mathbb{E}_x[\nabla_\varphi F]\otimes\mathbb{E}_x[\nabla_\varphi F]}{\mathbb{E}_x[F]^2} \\ &\qquad + \left(\frac{1}{2} + C_T(\mathrm{Rc}) + C_T(\mathrm{Rm},\nabla\mathrm{Rc})\frac{\mathbb{E}_x[F^2]^{1/2}}{\mathbb{E}_x[F]}\right)||\varphi||^2 g_x \geq 0, \end{aligned} \tag{11}$$

where $C_T(\mathrm{Rc}) < \infty$ and $C_T(\mathrm{Rm},\nabla\mathrm{Rc}) < \infty$ are constants, which converge to 0 as $|\mathrm{Rc}| \to 0$ and $|\mathrm{Rm}| + |\nabla\mathrm{Rc}| \to 0$, respectively.

References

[1] R. Hamilton, *A Matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. 1(1):113–126, 1993.

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[3] P. Li, S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156(3-4):153–201, 1986.