The bounded diameter conjecture for two-convex mean curvature flow

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(joint work with Panagiotis Gianniotis)

Given any closed embedded initial hypersurface $M_0^n \subset \mathbb{R}^{n+1}$, there exists a unique smooth solution $\{M_t^n \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ of the mean curvature flow defined on a maximal time interval $[0,T)$. One naturally wonders whether one can control the intrinsic diameter as one approaches the first singular time T. Another related question is whether one can obtaining sharp integral bounds for the mean curvature, e.g. it has been proved by Topping [8] that

$$\mathrm{diam}(M_t, d_t) \leq C_n \int_{M_t} |H|^{n-1}\, d\mu\,. \tag{1}$$

Note that for three-convex hypersurfaces, e.g. for $M_0 = S_r^{n-2}\times S_R^2$, it can happen that $\lim_{t\nearrow T}\int_{M_t} H^{n-1}\, d\mu = \infty$. We thus assume that M_0 is two-convex, i.e. that the sum of the smallest two principal curvatures is positive. It has been proved by Head [5] and Cheeger-Haslhofer-Naber [1] that for the mean curvature flow of two-convex hypersurfaces one has

$$\int_{M_t} |H|^{n-1-\varepsilon}\, d\mu \leq C(M_0,\varepsilon). \tag{2}$$

Motivated by this result, it is natural to conjecture:[1]

Conjecture (Bounded diameter conjecture)**.** *If $\{M_t \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ is a mean curvature flow of two-convex closed embedded hypersurfaces, then*

$$\mathit{diam}(M_t, d_t) \leq C(M_0). \tag{3}$$

Conjecture (L^{n-1}-curvature conjecture)**.** *If $\{M_t \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ is a mean curvature flow of two-convex closed embedded hypersurfaces, then*

$$\int_{M_t} H^{n-1}\, d\mu \leq C(M_0). \tag{4}$$

The question of whether or not one can actually get rid of the ε in estimates like (2) depends on the fine structure of singularities and high curvature regions. For comparison, in a recent breakthrough [6], Naber-Valtorta improved the known $L^{3-\varepsilon}$-estimates for the gradient of minimizing harmonic maps to sharp L^3_{weak}-estimates. The simple example $u(x) = x/|x|$ in dimension three, shows that in their case the L^3_{weak}-estimate actually *cannot* be replaced by an L^3-estimate.

In joint work with Panagiotis Gianniotis, we proved the above two conjectures. More precisely, we proved the following two theorems:

[1] I thank John Head for introducing me to these conjectures in 2011. While we unfortunately don't know the precise history, the conjectures have certainly been discussed among experts well before 2011, c.f. Perelman's bounded diameter conjecture for 3d Ricci flow [7].

Theorem (Intrinsic diameter bound [3])**.** *If* $\{M_t \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ *is a mean curvature flow of two-convex embedded hypersurfaces, then*

$$diam(M_t, d_t) \leq C, \tag{5}$$

for a constant $C = C(\mathcal{A}, \alpha, \beta, \gamma) < \infty$*, which only depends on certain geometric parameters of the initial hypersurface* M_0 *(area bound, noncollapsing constant, two-convexity constant, initial mean curvature bound).*

Theorem (Sharp curvature estimate [3])**.** *If* $\{M_t \subset \mathbb{R}^{n+1}\}_{t\in[0,T)}$ *is a level set flow with smooth two-convex initial data, then we have the sharp estimate*

$$\int_{M_t} H^{n-1} d\mu \leq C, \tag{6}$$

where $C = C(\alpha, \beta, \gamma, \mathcal{A}) < \infty$ *only depends on the geometry of* M_0*.*

Our proofs rely on a detailed analysis of cylindrical regions (ε-tubes) under mean curvature flow. In particular, we use the Łojasiewicz inequality from Colding-Minicozzi [2] and the canonical neighborhood theorem from Haslhofer-Kleiner [4].

References

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