Weak solutions for the Ricci flow

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(joint work with Aaron Naber)

We introduce a new class of estimates for the Ricci flow, and use them both to characterize solutions of the Ricci flow and to provide a notion of weak solutions for the Ricci flow in the nonsmooth setting.

As a motivation, let us first explain the much easier task of characterizing supersolutions of the Ricci flow. Let $(M, g_t)_{t\in I}$ be a one-parameter family of Riemannian manifolds. We consider the heat equation $(\partial_t - \Delta_{g_t})w = 0$ on our evolving manifolds $(M, g_t)_{t\in I}$. For every $s, T \in I$ with $s \leq T$, and every smooth function u with compact support, we write $P_{sT}u$ for the solution at time T with initial condition u at time s, i.e. $(P_{sT}u)(x) = \int_M u(y)\, H(x, T\,|\,y, s) dV_s(y)$, where $H(x, T\,|\,y, s)$ is the heat kernel with pole at (y, s). We write $d\nu_{(x,T)}(y, s) = H(x, T\,|\,y, s)dV_s(y)$.

Proposition ([1]). *The following are equivalent:*

(1) $\partial_t g_t \geq -2Rc_{g_t}$
(2) $|\nabla P_{sT}u| \leq P_{sT}|\nabla u|$
(3) $|\nabla P_{sT}u|^2 \leq P_{sT}|\nabla u|^2$
(4) $\int_M u^2 \log u^2\, d\nu \leq 4(T-s)\int_M |\nabla u|^2 d\nu$
(5) $\int_M (u-\bar{u})^2\, d\nu \leq 2(T-s)\int_M |\nabla u|^2 d\nu$.

In essence, the proposition follows easily from the parabolic Bochner-formula

$$(\partial_t - \Delta)|\nabla u|^2 = 2\langle \nabla u, \nabla(\partial_t - \Delta)u\rangle - 2|\nabla^2 u|^2 - (\partial_t g + 2\mathrm{Rc})(\nabla u, \nabla u).$$

To characterize solutions of the Ricci flow, and not just supersolutions, we prove infinite-dimensional generalizations of the above estimates. Let $(M, g_t)_{t\in I}$ be a smooth family of Riemannian manifolds. Let $\mathcal{M} = M \times I$ be its space-time with the usual space-time connection, i.e. $\nabla_t Y = \partial_t Y + \frac{1}{2}\partial_t g_t(Y, \cdot)^{\sharp_{g_t}}$. For each $(x, T) \in \mathcal{M}$, we consider the based path space $P_{(x,T)}\mathcal{M}$ consisting of all space-time curves of the form $\{\gamma_\tau = (x_\tau, T-\tau)\}_{\tau\in[0,T]}$, where $\{x_\tau\}_{\tau\in[0,T]}$ is a continuous curve in M with $x_0 = x$. Let $\Gamma_{(x,T)}$ be the Wiener measure of Brownian motion on our evolving family of manifolds based at (x, T), i.e. the probability measure uniquely characterized by the following property. If $e_{\sigma_1,\ldots,\sigma_k} : P_{(x,T)}\mathcal{M} \to M^k$, $\gamma \mapsto (x_{\sigma_1}, \ldots, x_{\sigma_k})$, is the evaluation map at $0 \leq \sigma_1 \leq \ldots \leq \sigma_k \leq T$ then

$$e_{\sigma_1,\ldots,\sigma_k *} d\Gamma_{(x,T)}(y_1, \ldots, y_k) = d\nu_{(x,T)}(y_1, s_1)\cdots d\nu_{(y_{k-1},s_{k-1})}(y_k, s_k),$$

where $s_i = T - \sigma_i$. Path space can be equipped with two natural notions of gradient, the parallel gradient $\nabla^{\|}$ and the Malliavin gradient $\nabla^{\mathcal{H}}$, see [1]. Our main theorem characterizes solutions of the Ricci flow in terms of certain sharp estimates on path space.

Theorem ([1]). *The following are equivalent:*

(1) $\partial_t g_t = -2Rc_{g_t}$
(2) $|\nabla_x \int_{P_T\mathcal{M}} F d\Gamma_{(x,T)}| \leq \int_{P_T\mathcal{M}} |\nabla^{\|} F|\, d\Gamma_{(x,T)}$
(3) $\int_{P_T\mathcal{M}} \frac{d[F^\bullet]_\tau}{d\tau} d\Gamma_{(x,T)} \leq 2\int_{P_T\mathcal{M}} |\nabla^{\|}_\tau F|^2\, d\Gamma_{(x,T)}$

(4) $\int_{P_T\mathcal{M}}(F^2)^{\tau_2}\log{(F^2)^{\tau_2}}-(F^2)^{\tau_1}\log{(F^2)^{\tau_1}}d\Gamma_{(x,T)} \leq 4\int_{P_T\mathcal{M}}\langle F, \mathcal{L}_{\tau_1,\tau_2}F\rangle d\Gamma_{(x,T)}$
(5) $\int_{P_T\mathcal{M}}(F^{\tau_2}-F^{\tau_1})^2\, d\Gamma_{(x,T)} \leq 2\int_{P_T\mathcal{M}}\langle F, \mathcal{L}_{\tau_1,\tau_2}F\rangle\, d\Gamma_{(x,T)}$

Here, F^τ denotes the martingale induced by $F\in L^2(P_T\mathcal{M},\Gamma_{(x,T)})$, and $\mathcal{L}_{\tau_1,\tau_2}$ denotes the $[\tau_1,\tau_2]$-part of the Ornstein-Uhlenbeck operator $\mathcal{L}=\nabla^{\mathcal{H}*}\nabla^{\mathcal{H}}$. The estimates from the theorem are infinite-dimensional generalizations of the estimates from the proposition. In the very special case of 1-point test functions, i.e. test functions of the form $F(\gamma)=u(\gamma(t_0))$ for some $u:M\to\mathbb{R}$, our infinite dimensional estimates reduce to the finite-dimensional estimates from the proposition. Of course, there are many more test functions on path space, and this is one of the reasons why our infinite-dimensional estimates are strong enough to characterize solutions of the Ricci flow, and not just supersolutions.

Finally, let us briefly indicate how the above characterization of solutions of the Ricci flow can be used to provide a notion of weak solutions for the Ricci flow [2]. We consider metric-measure spaces $\mathcal{M}$ equipped with a time function and a linear heat flow. We call $\mathcal{M}$ a weak solution of the Ricci flow if and only if the infinite dimensional gradient estimate $|\nabla_x\int_{P_T\mathcal{M}}F d\Gamma_{(x,T)}|\leq\int_{P_T\mathcal{M}}|\nabla^{\|}F|\,d\Gamma_{(x,T)}$ holds. We establish various geometric and analytic estimates for these weak solutions. In particular, one of our applications concerns a question of Perelman about limits of Ricci flows with surgery [4]. Namely, the metric completion of the space-time of Kleiner-Lott [3], which they obtained as a limit of Ricci flows with surgery where the neck radius is sent to zero, is a weak solution in our sense.

References

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