SOME RECENT APPLICATIONS OF MEAN CURVATURE FLOW WITH SURGERY

ROBERT HASLHOFER

Abstract. In this note we survey some recent topological, geometric and analytic applications of mean curvature flow with surgery. This is based on joint projects with Buzano-Hershkovits, Ketover, and Ivaki.

The purpose of this note is to survey some recent applications of mean curvature flow with surgery. Recall that a flow with surgery is a smooth mean curvature flow interrupted by finitely many surgeries, where certain necks are replaced by standard caps and connected components covered entirely by canonical neighborhoods are discarded. By work of Huisken-Sinestrari [13], Brendle-Huisken [2] and Haslhofer-Kleiner [11] mean curvature flow with surgery exists for any mean convex surface in a three-manifold and for any two-convex hypersurface in Euclidean space of arbitrary dimension.

1. Topological applications

Consider the moduli space of embedded n-spheres in $\mathbb{R}^{n+1}$, i.e. the space

$$\mathcal{M}(S^n) = \mathrm{Emb}(S^n, \mathbb{R}^{n+1})/\mathrm{Diff}(S^n) \tag{1.1}$$

equipped with the smooth topology. By a classical theorem of Smale the space $\mathcal{M}(S^1)$ is contractible [19], and by Hatcher's solution of the Smale conjecture $\mathcal{M}(S^2)$ is also contractible [12]. For $n \geq 3$, there are many non-vanishing homotopy groups, see e.g. [6]. In the view of the topological complexity of $\mathcal{M}(S^n)$ for general n, it is an interesting question whether one can still derive some positive results on the space of embedded n-spheres under some curvature conditions. Motivated by the topological classification result from [13], we consider the subspace $\mathcal{M}_{\text{2-conv}}(S^n) \subset \mathcal{M}(S^n)$ of two-convex embedded n-spheres in $\mathbb{R}^{n+1}$, i.e. we impose the condition that the sum of the two smallest principal curvatures is positive. We proved:

Theorem 1.1 (Buzano-Haslhofer-Hershkovits [4]). *The space $\mathcal{M}_{\text{2-conv}}(S^n)$ is path-connected in every dimension n.*

Our proof uses mean curvature flow with surgery, a connected sum operation for two-convex hypersurfaces, and a scheme inspired by the work

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of Marques on the moduli space of positive scalar curvature metrics on the three-sphere [16]. The gist of the proof is illustrated in Figure 1.

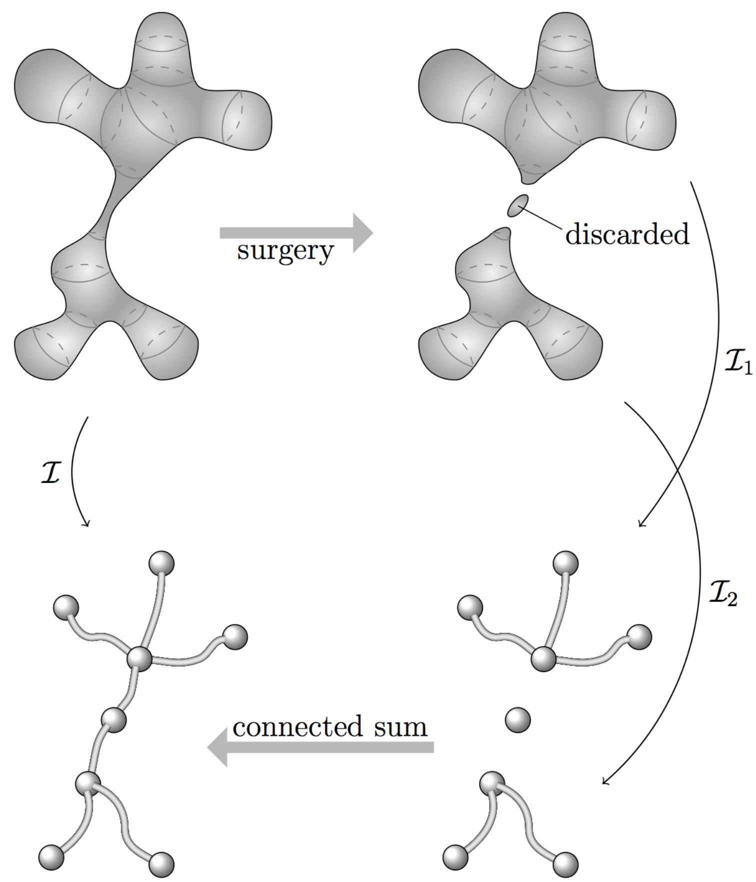

FIGURE 1. By inductive assumption, each connected component of the post-surgery manifold and every discarded component is isotopic to a marble tree. We prove that these isotopies can be glued together to yield an isotopy of the pre-surgery manifold to a connected sum of the marble trees.

Extending our method to the case of tori, we considered the moduli space of two-convex embedded tori $\mathcal{M}_{2-\mathrm{conv}}(S^{n-1}\times S^1)$ and proved:

Theorem 1.2 (Buzano-Haslhofer-Hershkovits [3]). *For $n\geq 3$ the space $\mathcal{M}_{2-conv}(S^{n-1}\times S^1)$ is path-connected, while for $n=2$ we have that two mean-convex embedded tori are in the same path-component of their moduli space if and only if they have the same knot class.*

For higher genus, there is the following recent result by Mramor:

Theorem 1.3 (Mramor [17]). *Any two-convex hypersurface can be isotoped monotonically to a marble graph. In particular, the collection of two-convex* *embedded hypersurfaces with controlled diameter, mean curvature and non-collapsing constant has finitely many isotopy classes.*

2. Geometric applications

To motivate the geometric applications, recall that by a classical theorem of Lusternik-Schnirelmann [15, 8] any closed Riemannian two-sphere contains at least three embedded closed geodesics. Moving up one dimension, it is conjectured that on the three-sphere S^3 equipped with any Riemannian metric g one can find at least four embedded minimal two-spheres. The existence of at least one minimal two-sphere was obtained by Simon-Smith in 1983 using min-max theory. Using combined efforts from min-max theory and mean curvature flow with surgery we proved the existence of at least one additional embedded minimal two-sphere besides the one from Simon-Smith:

Theorem 2.1 (Haslhofer-Ketover [10]). *Let M be a three-manifold diffeomorphic to S^3 and endowed with a bumpy metric. Then M contains at least two embedded minimal two-spheres. More precisely, exactly one of the following alternatives holds:*

(1) *M contains at least one stable embedded minimal two-sphere, and at least two embedded minimal two-spheres of index one.*
(2) *M contains no stable embedded minimal two-sphere, at least one embedded minimal two-sphere Σ_1 of index one, and at least one embedded minimal two-sphere Σ_2 of index two. In this case*

$$Area(\Sigma_2) < 2Area(\Sigma_1).$$

In the easier case (1) the manifold M looks like a kind of dumbbell, and two embedded minimal two-spheres of index one can be obtained by considering one-parameter sweepouts of both halves. To illustrate case (2), given $a > b > c > d > 0$, consider the ellipsoid

$$E(a, b, c, d) := \left\{ \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1 \right\} \subset \mathbb{R}^4. \tag{2.1}$$

It contains at least 4 minimal 'planar' two-spheres, which are obtained by the intersection of $E(a, b, c, d)$ with the coordinates hyperplanes $\{x_i = 0\}$. S.T. Yau asked (c.f. Section 4 in [20]) if the only minimal two-spheres in an ellipsoid centered about the origin in $\mathbb{R}^4$ are the planar ones. We obtain the following negative answer to Yau's question:

Corollary 2.2 (Haslhofer-Ketover [10]). *For fixed b, c and d, if a is chosen sufficiently large, then the ellipsoid $E(a, b, c, d)$ contains a non-planar embedded minimal two-sphere.*

Let us now sketch the proof of Theorem 2.1 in case (2). Using Simon-Smith's existence theorem we obtain one embedded minimal two-sphere of index one. Sliding the Simon-Smith sphere a bit to both sides we can decompose $M = D_1 \cup Z \cup D_2$ where Z is the short cylindrical region obtained by sliding the Simon-Smith sphere around, and D_1 and D_2 are smooth embedded 3-discs with mean convex boundary. To proceed, we prove the following general theorem establishing the existence of smooth mean convex foliations in three-manifolds:

Theorem 2.3 (Haslhofer-Ketover [10]). *Let $D \subset M^3$ be a smooth three-disc with mean convex boundary. Then exactly one of the following alternatives holds true:*

(1) *There exists an embedded stable minimal two-sphere $\Sigma \subset Int(D)$.*
(2) *There exists a smooth foliation $\{\Sigma_t\}_{t\in[0,1]}$ of D by mean convex embedded two-spheres.*

Note that in our application we are always in case (2). We have stated Theorem 2.3 in its more general form since it is clearly of independent interest and also has other applications, see e.g. [1]. To produce the smooth mean convex foliation in Theorem 2.3 we use mean curvature flow with surgery and a variant of the argument from the proof of Theorem 1.1.

Having discussed Theorem 2.3, let us now sketch how it can be used to finish the proof of Theorem 2.1. Recalling that $M = D_1 \cup Z \cup D_2$ and using the foliations of D_1 and D_2 produced by Theorem 2.3 we can build an optimal foliation of M, by which we mean a foliation $\{\Sigma_t\}_{t\in[-1,1]}$ of M by two-spheres so that the Simon-Smith sphere sits in the middle of the foliation as Σ_0 and all other slices have less area. From the one-parameter family $\{\Sigma_t\}$ we can then form a two-parameter family $\{\Sigma_{s,t}\}$. Loosely speaking, $\Sigma_{s,t}$ consist of the two surfaces Σ_s and Σ_t joined by a very thin tube. Using the catenoid estimate from Ketover-Marques-Neves [14] we show that

$$\sup_{s,t} \mathrm{Area}(\Sigma_{s,t}) < 2\mathrm{Area}(\Sigma_0). \tag{2.2}$$

The bound (2.2) guarantees that the minimal surface produced from two-parameter min-max is not simply twice the minimal sphere Σ_0. Finally, by Lusternik-Schnirelmann theory, the minimal surface produced cannot be Σ_0 with multiplicity one.

3. Analytic applications

For the analytic applications we consider the Allen-Cahn equation

$$\Delta_g u = \frac{1}{\varepsilon^2} u(1-u^2) \tag{3.1}$$

on three-spheres. The beautiful work of Gaspar-Guaraco [7] establishes the existence of solutions of arbitrarily large index as ε becomes smaller. We addressed the remaining question of finding solutions with low complexity, i.e. solutions with interface of genus zero and low index. Before stating the theorem let us clarify that nontrivial solutions always come in pairs, i.e., whenever u is a solution then $-u$ is also a solution. We proved:

Theorem 3.1 (Haslhofer-Ivaki [9]). *Consider the Allen-Cahn equation* (3.1) *on* S^3 *endowed with any bumpy metric* g*. Then for any small enough* $\varepsilon > 0$ *there exist at least two pairs* $\{u_\varepsilon^1, -u_\varepsilon^1\}, \{u_\varepsilon^2, -u_\varepsilon^2\}$ *of solutions with spherical interface and index at most two. More precisely, exactly one of the following alternatives occurs:*

(1) *At least one pair of stable solutions with spherical interface, and at least two pairs of index-one solutions with spherical interface.*
(2) *No stable solutions with spherical interface, at least one pair of index-one solutions with spherical interface, and at least one pair of index-two solutions with spherical interface.*

The solutions transition from $u_\varepsilon \approx -1$ to $u_\varepsilon \approx +1$ along the spherical interface, and the transition is modeled on the function $\tanh(\frac{t}{\sqrt{2}\varepsilon})$, where t denotes the signed distance from the interface. Our proof is based on combining several results from the literature, including in particular the gluing construction from Pacard-Ritore [18], the existence result for minimal two-spheres described in the previous section, and the index estimate from Chodosh-Mantoulidis [5]. Finally, we can also obtain further solutions by gluing together one, two or three of the solutions from case (1):

Corollary 3.2 (Haslhofer-Ivaki [9]). *If the Allen-Cahn equation* (3.1) *on* S^3 *endowed with a bumpy metric* g *has a stable solution with spherical interface, then there are at least seven pairs of solutions with index at most two.*

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Department of Mathematics, University of Toronto, 40 St George Street, Toronto, ON M5S 2E4, Canada

E-mail address: roberth@math.toronto.edu