A NONEXISTENCE RESULT FOR WING-LIKE MEAN CURVATURE FLOWS IN $\mathbb{R}^4$

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Abstract. Some of the most worrisome potential singularity models for the mean curvature flow of 3-dimensional hypersurfaces in $\mathbb{R}^4$ are noncollapsed wing-like flows, i.e. noncollapsed flows that are asymptotic to a wedge. In this paper, we rule out this potential scenario, not just among self-similarly translating singularity models, but in fact among all ancient noncollapsed flows in $\mathbb{R}^4$. Specifically, we prove that for any ancient noncollapsed mean curvature flow $M_t = \partial K_t$ in $\mathbb{R}^4$ the blowdown $\lim_{\lambda\to 0} \lambda \cdot K_{t_0}$ is always a point, halfline, line, halfplane, plane or hyperplane, but never a wedge. In our proof we introduce a fine bubble-sheet analysis, which generalizes the fine neck analysis that has played a major role in many recent papers. Our result is also a key first step towards the classification of ancient noncollapsed flows in $\mathbb{R}^4$, which we will address in a series of subsequent papers.

Contents

1. Introduction

The mean curvature flow of mean-convex surfaces in $\mathbb{R}^3$, and also of 2-convex hypersurfaces in $\mathbb{R}^{n+1}$, is by now well understood. In particular, White proved that the evolving surfaces are smooth away from a small set of times [Whi00, Whi03], and Huisken-Sinestrari [HS09], Brendle-Huisken [BH16] and Haslhofer-Kleiner [HK17b] constructed a flow with surgery. More recently, in significant work by Angenent-Daskalopoulos-Sesum [ADS19, ADS20] and Brendle-Choi [BC19, BC] a complete classification of all possible singularity models for such flows has been obtained (these classification results have in turn be generalized in our recent proof of the mean-convex neighborhood conjecture [CHH18, CHHW19]).

On the other hand, the singularity formation for the flow of 3-dimensional hypersurfaces in $\mathbb{R}^4$ is much less understood. In particular, one of the most worrisome potential scenarios is that one encounters so-called wing-like flows as blowup limits, i.e. flows that are asymptotic to a wedge.

To explain the background, let us first discuss the situation of 2-dimensional flows in $\mathbb{R}^3$, and in fact let us restrict the discussion even further to the case of strictly convex, graphical, self-similarly translating solutions. The most well known such solution is the translating bowl soliton [AW94], which is given by a rotationally symmetric entire graph. However, in addition to the bowl there is also a one-parameter family of solutions defined over strip regions $(-b, b) \times \mathbb{R}$ for every $b > \pi/2$. These solutions have been constructed by Ilmanen (unpublished) and Wang [Wan11], and have been classified in recent work by Hoffman-Ilmanen-Martin-White [HIMW18], building also on important prior work by Spruck-Xiao [SX17]. Ilmanen called them Δ-wings, capturing their shape. They are asymptotic to a wedge with its sides modelled on the grim-reaper times $\mathbb{R}$.

The 2-dimensional Δ-wings do not cause any deep concerns for the singularity analysis of mean curvature flow. This is because they are collapsed. In fact, it is known that collapsed solutions cannot arise as blowup limit of any mean-convex flow [Whi00, Whi03] (see also [SW09, And12, HK17a]), and conjectured that they cannot arise as blowup limit of any embedded flow [Ilm95]. We recall that a mean-convex flow is called noncollapsed, if there is some $\alpha > 0$ such that every point admits interior and exterior balls of radius at least $\alpha/H(p)$.

However, the situation changes dramatically when one moves one dimension higher. This is because the grim-reaper is collapsed, but the bowl is noncollaped. Hence, one has to worry about the potential scenario that a blowup limit could be a wing-like translator with its sides modelled on the 2-dimensional bowl times $\mathbb{R}$. This is illustrated in Figure 1. More generally, this lead to the question:

Question 1.1 (noncollapsed wing-like translators)**.** *Do there exist any noncollapsed wing-like translators in $\mathbb{R}^4$?*

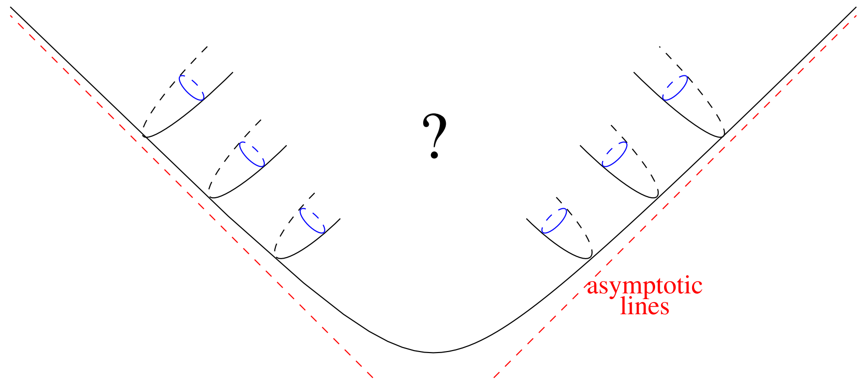

FIGURE 1. One of the most worrisome potential singularity models would be a three-dimensional noncollapsed wing-like flow in $\mathbb{R}^4$ with its sides modelled on the 2-dimensional bowl times $\mathbb{R}$.

Here, with wing-like we mean any solution that is asymptotic to a wedge, i.e. a 2-dimensional convex cone with angle strictly less than π. Such solutions would cause a major complication for understanding the flow through singularities in $\mathbb{R}^4$, and would also be a major obstacle for the construction of a flow with surgery. More generally, one also has to worry about blowup limits that are not necessarily self-similar:

Question 1.2 (noncollapsed wing-like ancient flows)**.** *Do there exist any noncollapsed wing-like ancient flows in* $\mathbb{R}^4$*?*

1.1. **Main results.** To address Question 1.1 and Question 1.2 (and even to properly define the notion of wing-like), we study the blowdown of the time slices. To this end, let $M_t = \partial K_t \subset \mathbb{R}^4$ be a noncollapsed ancient solution of the mean curvature flow. We recall that such flows are always convex thanks to [HK17a, Theorem 1.10].

Definition 1.3 (blowdown)**.** Given any time t_0, the *blowdown* of K_{t_0} is defined by

$$\check{K}_{t_0} := \lim_{\lambda \to 0} \lambda \cdot K_{t_0}. \tag{1}$$

For example, if the flow is a three-dimensional bowl soliton, then its blowdown is a halfline. On the other hand, if the flow is a 2-dimensional bowl soliton times $\mathbb{R}$, then its blowdown is a 2-dimensional halfplane. The scenario of a wing-like flow would correspond to the case where the blowdown is a wedge, i.e. a 2-dimensional convex cone with angle strictly less than π.

Our main theorem provides a complete classification of all possible blowdowns of noncollapsed flows in $\mathbb{R}^4$:

Theorem 1.4 (blowdown of noncollapsed flows in $\mathbb{R}^4$)**.** *Let $M_t = \partial K_t$ be an ancient noncollapsed mean curvature flow in $\mathbb{R}^4$. Then its blowdown is*

- *either a point (which only happens if the solution is compact),*
- *or a halfline,*
- *or a line (which only happens for $S^2 \times \mathbb{R}$ or oval times $\mathbb{R}$),*
- *or a 2-dimensional halfplane (which only happens if the solution is a 2-dimensional bowl soliton times $\mathbb{R}$),*
- *or a 2-dimensional plane (which only happens for $S^1 \times \mathbb{R}^2$),*
- *or a 3-dimensional hyperplane (which only happens for flat $\mathbb{R}^3$).*

In particular, the blowdown can never be a wedge.

Theorem 1.4 shows that the asymptotic structure of noncollapsed flows, i.e. the flows that are actually relevant for singularity analysis, is much more rigid than the asymptotic structure of arbitrary convex flows. In fact, the example of the Δ-wings illustrates that even for 2-dimensional translators in $\mathbb{R}^3$ one can get the wedge of any angle less than π as blowdown. In $\mathbb{R}^4$ there is a zoo of collapsed ancient convex solutions, see e.g. [Wan11, HIMW18], whose blowdown can be much more arbitrary than the ones in Theorem 1.4.

As an immediate corollary, we can answer Question 1.2. Namely, since the blowdown can never be a wedge, we can rule out the potential scenario of noncollapsed wing-like ancient flows in $\mathbb{R}^4$:

Corollary 1.5 (Non-existence of wing-like noncollapsed flows)**.** *Wing-like noncollapsed ancient flows in $\mathbb{R}^4$ do not exist.*

In particular, this also answers Question 1.1:

Corollary 1.6 (Non-existence of wing-like noncollapsed translators)**.** *Wing-like noncollapsed translators in $\mathbb{R}^4$ do not exist.*

Corollary 1.5 and Corollary 1.6 rule out some of the most worrisome potential singularity models for the mean curvature flow of 3-dimensional mean-convex hypersurfaces in $\mathbb{R}^4$ (and more generally also for mean-convex flows in 4-manifolds, since after blowup the ambient manifold always becomes Euclidean).

1.2. **Outline and further results.** Let us now outline our approach. Let $M_t = \partial K_t$ be an ancient noncollapsed mean curvature flow in $\mathbb{R}^4$. We can assume that the solution is noncompact and nonflat, as otherwise the blowdown would simply be a point or a 3-dimensional hyperplane, respectively.

As we explain in Section 2, it follows from the general theory of noncollapsed flows [HK17a] that the blowdown

$$\check{K} := \lim_{\lambda \to 0} \lambda \cdot K_{t_0}. \tag{2}$$

always is a convex cone of dimension at most 2 (also, this limit is in fact independent of t_0). If $\check{K}$ splits off a line, then so does the flow $M_t = \partial K_t$, and we are done by the classification of 2-dimensional noncollapsed flows in $\mathbb{R}^3$ by Brendle-Choi [BC19] and Angenent-Daskalopoulos-Sesum [ADS20]. After these reductions, the task is thus to rule out the scenario where $\check{K}$ is wedge.

We next consider the tangent flow at $-\infty$. Flows with a *neck-tangent flow at* $-\infty$, i.e. with

$$\lim_{\lambda\to 0} \lambda M_{\lambda^{-2}t} = \mathbb{R}\times S^2(\sqrt{-4t}), \tag{3}$$

have already been classified by [BC]. We can thus assume that we have a *bubble-sheet tangent flow at* $-\infty$, i.e. that

$$\lim_{\lambda\to 0} \lambda M_{\lambda^{-2}t} = \mathbb{R}^2\times S^1(\sqrt{-2t}). \tag{4}$$

In Section 3, to facilitate various calibration and barrier arguments, we build a foliation for flows close to a bubble sheet in $\mathbb{R}^4$. We do this by shifting and rotating the 2-dimensional shrinker foliation in $\mathbb{R}^3$ from [ADS19].

In Section 4, we set up the fine bubble-sheet analysis, which generalizes the fine neck-analysis that played a major role in [ADS19, ADS20, BC19, BC, CHH18, CHHW19]. Given any space-time point X_0, we consider the renormalized flow

$$\bar{M}_\tau = e^{\frac{\tau}{2}}\left(M_{t_0-e^{-\tau}} - x_0\right). \tag{5}$$

Then, the hypersurfaces $\bar{M}_\tau$ converge for $\tau\to-\infty$ to the cylinder

$$\Gamma = \mathbb{R}^2\times S^1(\sqrt{2}). \tag{6}$$

In particular, $\bar{M}_\tau$ can be written as the graph of a function $u = u_{X_0}(\cdot,\tau)$ with small norm over $\Gamma\cap B_{2\rho(\tau)}$, where $\rho(\tau)\to\infty$ as $\tau\to-\infty$. The goal is then to derive a sharp asymptotic expansion for the graph function u.

The analysis over Γ is governed by the Ornstein-Uhlenbeck operator

$$\mathcal{L} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{1}{2}\frac{\partial^2}{\partial\theta^2} - \frac{1}{2}x_1\frac{\partial}{\partial x_1} - \frac{1}{2}x_2\frac{\partial}{\partial x_2} + 1. \tag{7}$$

The operator $\mathcal{L}$ has 5 unstable eigenfunctions, namely

$$1, \cos\theta, \sin\theta, x_1, x_2, \tag{8}$$

and 7 neutral eigenfunctions, namely

$$x_1\cos\theta, x_1\sin\theta, x_2\cos\theta, x_2\sin\theta, x_1^2-2, x_2^2-2, x_1x_2. \tag{9}$$

Using the ODE-lemma from Merle-Zaag [MZ98], we show that for $\tau\to-\infty$ either the unstable mode is dominant or the neutral mode is dominant. Furthermore, considering instead an expansion for

$$\tilde{M}_\tau = S(\tau)\bar{M}_\tau, \tag{10}$$

where the fine-tuning rotation $S(\tau)\in\mathrm{SO}(4)$ is obtained via the implicit function theorem, we can assume that in the neutral mode case the truncated function $\hat{u}$ is orthogonal to the θ-dependent functions from (9).

In Section 5, we consider the case where the neutral mode is dominant. By the reductions from above we can assume that the blowdown $\check{K}$ is a convex cone that does not contain any line. Furthermore, rotating coordinates, we can arrange that $\check{K}$ contains the positive x_1-axis, namely

$$\{\lambda e_1 \,|\, \lambda \geqslant 0\} \subseteq \check{K}. \tag{11}$$

In this setting, we have:

Theorem 1.7 (blowdown in neutral mode)**.** *If the neutral mode is dominant, then its blowdown is a halfline, namely*

$$\check{K} = \{\lambda e_1 \,|\, \lambda \geqslant 0\}. \tag{12}$$

Moreover,

$$\lim_{\tau\to-\infty} \frac{\hat{u}(\cdot,\tau)}{\|\hat{u}(\cdot,\tau)\|} = -c(x_2^2-2), \tag{13}$$

where $c > 0$.

To prove this, remembering (9) and the fine-tuning, we first show that along a subsequence we have

$$\lim_{\tau_{i_m}\to-\infty} \frac{\hat{u}(\cdot,\tau_{i_m})}{\|\hat{u}(\cdot,\tau_{i_m})\|} = q_{11}(x_1^2-2) + q_{22}(x_2^2-2) + 2q_{12}x_1x_2. \tag{14}$$

Using Brunn's concavity principle, we show that $Q = \{q_{ij}\}$ is a nontrivial semi-negative definite 2×2-matrix.
The crucial step is then to relate the algebra of the quadratic form Q with the geometry of the blowdown $\check{K}$. Specifically, we show that

$$\check{K} \subseteq \ker Q. \tag{15}$$

The basic idea for this is that in directions $v \notin \ker Q$ we have an inwards quadratic bending, which implies that v is a "short" direction, i.e. $v \notin \check{K}$. This can be made precise using again Brunn's concavity principle. Once (15) is established, Theorem 1.7 follows easily. In fact, our argument also gives a quantitative estimate for the diameter of the short directions:

Corollary 1.8 (diameter of level sets)**.** *If $M_t = \partial K_t$ is as above, then for any $\delta > 0$ we have*

$$\bar{M}_\tau \cap \{x_1 = 0\} \subseteq B_{e^{\delta|\tau|}}(0)$$

for $\tau \ll 0$.

In Section 6, we consider the case where the unstable mode is dominant. Our main technical result for such flows is the following:

Theorem 1.9 (Fine bubble-sheet theorem)**.** *Let $\{M_t\}$ be an ancient noncollapsed flow in $\mathbb{R}^4$, with a bubble sheet tangent flow at $-\infty$, whose unstable mode is dominant. Then there exist universal constants a_1, a_2, such that for every X, after suitable re-centering in the x_3x_4-plane, the truncated graph function $\check{u}^X(\cdot,\tau)$ of the renormalized flow $\bar{M}^X_\tau$ satisfies*

$$\check{u}^X = e^{\frac{\tau}{2}}\,(a_1x_1 + a_2x_2) + o(e^{\frac{\tau}{2}}) \tag{16}$$

for $\tau \ll 0$ depending only on an upper bound on the bubble-sheet scale $Z(X)$. Moreover, the expansion parameters satisfy

$$|a_1| + |a_2| > 0. \tag{17}$$

Theorem 1.9 shows that the bubble-sheet increases its bubble-size slightly, in a precisely described way, as one moves in direction of the vector (a_1, a_2). The constants a_1, a_2 are genuine constants of the flow. For example, if M_t is $\mathbb{R}$ times a 2-dimensional bowl translating in x_2-direction, then $a_1 = 0$ and a_2 is proportional to the reciprocal of the speed of translation.

The fine-bubble sheet theorem (Theorem 1.9) generalizes the fine-neck theorem from our prior work [CHH18, CHHW19]. To prove it, we follow the scheme from our prior work with the necessary modifications. In particular, we now expand in terms of the unstable eigenfunctions from (8), and we use the new foliation from Section 3. Another new step is to show that the unstable mode is dominant even after removing the fine-tuning rotation.

In Section 7, we play the following end game. Let $M_t = \partial K_t$ be a noncompact ancient noncollapsed flow in $\mathbb{R}^4$, with bubble-sheet tangent flow at $-\infty$, whose unstable mode is dominant, and suppose towards a contradiction that its blowdown is a wedge. Then taking suitable limits along the sides, we see $\mathbb{R}$ times a 2-dimensional bowl translating soliton. As observed above, the vector (a_1, a_2) points in the translation direction. However, since the limits obtained along the two different sides of the wedge translate in different directions this gives the desired contradiction. In fact, the argument also works for a degenerate wedge (i.e. a halfline) and thus proves:

Theorem 1.10 (classification in unstable mode)**.** *The only noncompact ancient noncollapsed flow in $\mathbb{R}^4$, with bubble-sheet tangent flow at $-\infty$, whose unstable mode is dominant, is $\mathbb{R} \times$ 2d-bowl.*

Theorem 1.10, in addition to ruling out the wedge blowdown, in fact completes the classification of noncompact ancient noncollapsed flows in $\mathbb{R}^4$ in case the unstable mode is dominant. We will address the neutral mode case in subsequent work, based on Theorem 1.7 and Corollary 1.8.

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2. Coarse properties of ancient noncollapsed flows

Recall that by [HK17a, Thm. 1.14] and [CM15], if $M_t = \partial K_t$ is a noncollapsed mean curvature flow in $\mathbb{R}^4$, that is noncompact and nonflat, then, up to rotation, either

$$\lim_{\lambda\to 0} \lambda K_{\lambda^{-2}t} = \mathbb{R}\times D^3(\sqrt{-4t}), \tag{18}$$

or

$$\lim_{\lambda\to 0} \lambda K_{\lambda^{-2}t} = \mathbb{R}^2\times D^2(\sqrt{-2t}). \tag{19}$$

In the case (18) we say that M_t has a *neck tangent flow at* $-\infty$, whereas in case (19) we say that M_t has a *bubble-sheet tangent flow at* $-\infty$.

In the neck tangent case, it follows from Brendle-Choi [BC] (or alternatively from [CHHW19]) that M_t is either a round shrinking $S^2\times\mathbb{R}$ or a 3-dimensional bowl. Hence, we can focus on the bubble-sheet case.

2.1. The blowdown of time slices.

In this section, we establish several elementary properties of the blowdown of time slices (Definition 1.3).

Proposition 2.1 (basic properties of blowdown)**.** *For any ancient noncollapsed mean curvature flow $M_t = \partial K_t$ in $\mathbb{R}^4$, that is noncompact and nonflat, the blowdown $\check{K}_{t_0}$ is a convex cone of dimension at most* 2*.*

Proof. Assume $t_0 = 0$ and $0\in M_0$ for ease of notation. By convexity (see [HK17a, Theorem 1.10]), we have $\lambda\cdot K_0\subseteq \mu\cdot K_0$ whenever $\lambda\leqslant\mu$. Hence,

$$\check{K}_0 := \lim_{\lambda\to 0}\lambda\cdot K_0 = \bigcap_{\lambda>0}\lambda\cdot K_0 \tag{20}$$

exists and is convex. Since K_0 is noncompact, $\check{K}_0$ contains a halfline. Clearly, we have $\lambda\cdot\check{K}_0 = \check{K}_0$ for any $\lambda > 0$, i.e. $\check{K}_0$ is a cone.
As the flow is moving inwards, for every $t < 0$ we have

$$\lambda K_0 \subseteq \lambda K_{\lambda^{-2}t}. \tag{21}$$

Thus,

$$\check{K}_0 = \lim_{\lambda\to 0} K_0 \subseteq \lim_{\lambda\to 0}\lambda K_{\lambda^{-2}t}, \tag{22}$$

where the limit in the right hand side is described (after suitable rotation) either by (18) or by (19). In either case, we infer that

$$\check{K}_0 \subseteq \mathbb{R}^2\times\{0\}, \tag{23}$$

so $\check{K}_0$ is at most 2-dimensional. This proves the proposition. □

Proposition 2.2 (dimension reduction). *Let $M_t = \partial K_t$ be an ancient noncollapsed mean curvature flow in $\mathbb{R}^4$. If $\check{K}_{t_0}$ contains a line, then M_t is either a static hyperplane, a round shrinking $\mathbb{R} \times S^2$, a round shrinking $\mathbb{R}^2 \times S^1$, a 2-dimensional bowl times $\mathbb{R}$, or a 2-dimensional ancient oval times $\mathbb{R}$.*

Proof. If $\check{K}_{t_0}$ contains a line, then so does K_{t_0}. Hence, the flow $M_t = \partial K_t$ splits off an $\mathbb{R}$-factor. By the classification of 2-dimensional noncollapsed flows in $\mathbb{R}^3$ by Angenent-Daskalopoulos-Sesum [ADS19] and Brendle-Choi [BC19], this implies the assertion. □

We will now show that the blowdown $\check{K}_{t_0}$ is independent of $t_0 < T_{\text{ext}}(\mathcal{M})$, where $T_{\text{ext}}(\mathcal{M})$ denotes the extinction time of our flow. To show this, we need the following lemma (we write ν for the outward unit normal).

Lemma 2.3 (interior balls). *For every $\theta_0 > 0$ and $H_0 < \infty$ there exists some $\delta = \delta(H_0, \theta_0) > 0$ with the following significance: If $p \in M_{t_0}$ is such that $H(p) \leqslant H_0$ and $\langle -\nu(p), v\rangle \geqslant \theta_0$ for every $0 \neq v \in \check{K}_{t_0}$, then we have $B(p+v, \delta) \subseteq K_{t_0}$ for every $v \in \check{K}_{t_0}$ with $||v|| \geqslant 1$.*

Proof. First, note that as $\lim_{\lambda\to 0} \lambda K_{t_0} = \lim_{\lambda \to 0} \lambda(K_{t_0} - p)$, we have $p + sv \in K_{t_0}$ for every $v \in \check{K}_{t_0}$ and $s \geqslant 0$. The convexity of K_{t_0} implies that the function $(0,\infty) \ni s \mapsto \text{dist}(p+sv, M_{t_0})$ is concave and positive on K_{t_0}, and thus, nondecreasing. Taking an interior tangent ball at p of radius α/H_0 yields the desired result. □

Proposition 2.4 (time-independence of blowdown). *The blowdown $\check{K}_{t_0}$ is independent of $t_0 < T_{\text{ext}}(\mathcal{M})$.*

Proof. It suffices to prove this in case $M_t = \partial K_t$ in $\mathbb{R}^4$ is noncompact and strictly convex. Fix some t_0 and consider the set I of all $t \in [t_0, \mathrm{T}_{\text{ext}}(\mathcal{M}))$ such that $\check{K}_{t_0} = \check{K}_t$. Since $K_{t_2} \subseteq K_{t_1}$ for $t_2 \geqslant t_1$ we clearly have that $\check{K}_{t_2} \subseteq \check{K}_{t_1}$ for $t_2 \geqslant t_1$, so I is a (potentially degenerate) subinterval of $[t_0, \mathrm{T}_{\text{ext}}(\mathcal{M}))$ that contains t_0. Letting $t_1 \in I$, and taking any $p \in K_{t_1}$, strict convexity implies that

$$\inf_{v \in \check{K}_{t_1} - \{0\}} \langle -\nu(p), v\rangle > 0. \tag{24}$$

By the lemma above, there exists some $\delta > 0$ such that for every $||v||| \geqslant 1$, $B(p+v, \delta) \subseteq K_{t_1}$. Therefore, by avoidance, $p + v \in K_t$ for all t with $|t - t_1| \leqslant \delta^2/6$, so all such t are also in I.
Thus, if $I \neq [t_0, T_{\text{ext}}(\mathcal{M}))$ then $I = [t_0, t_1)$ for some $t_1 < T_{\text{ext}}(\mathcal{M})$. We want to show that this is impossible. Suppose, for the sake of contradiction, that there exists some $v \in \check{K}_{t_0} \backslash \check{K}_{t_1}$. Given any $p \in M_{t_1}$, by smoothness, we can find $t \in I$ arbitrarily close to t_1 and $p_t \in M_t$ such $p_t \to p$ and such that $H(p_t) \leqslant H(p) + 1$. As above, Lemma 2.3 and avoidance imply that

$$\langle \nu(p_t), v\rangle \to 0, \tag{25}$$

so $v \in T_p M_{t_1}$. But since $p \in M_{t_1}$ was arbitrary, this implies that M_{t_1} splits a line in the direction v, contradicting the strict convexity of M_{t_1}. □

We end this section by giving a (well-known) description of the blowdown in terms of graphical directions:

Proposition 2.5 (alternative description of blowdown)**.** *Let $K_t = \partial M_t$ be a noncollapsed strictly convex flow. Then*

$$\check{K}_{t_0} = \{\omega \in \mathbb{R}^{n+1} \text{ s.t } \langle \nu(p), \omega \rangle < 0 \text{ for every } p \in M_{t_0}\} \cup \{0\}. \tag{26}$$

Proof. Let $\Omega = \{\omega \in \mathbb{R}^{n+1} \text{ s.t } \langle \nu(p), \omega \rangle < 0 \text{ for every } p \in M_{t_0}\}$. The containment $\check{K}_{t_0} \subseteq \Omega \cup \{0\}$ is clear, as for every $p \in M_{t_0}$ and $v \in \check{K}_{t_0}$ $p + sv \in K_{t_0}$ for every positive s. Conversely, if $v \in \Omega$, for every $p \in M_{t_0}$ we have that $p + sv \in K_{t_0}$ for small values of s. Thus, if $p + s_0 v \in M_{t_0}$ for some $s_0 > 0$ then for $s > s_0$ with $s - s_0$ sufficiently small $p + sv \in K_{t_0}$, which contradicts the concavity of the distance to the boundary. Thus, the entire line $\{p + sv \mid s \in [0, \infty)\}$ is contained in K_{t_0}, so $v \in \check{K}_{t_0}$. □

2.2. **The bubble-sheet scale.** Let $\mathcal{M}$ be an ancient mean curvature flow, with a bubble-sheet tangent flow at $-\infty$. Given a point $X = (x, t) \in \mathcal{M}$ and a scale $r > 0$, we consider the flow

$$\mathcal{M}_{X,r} = \mathcal{D}_{1/r}(\mathcal{M} - X), \tag{27}$$

which is obtained from $\mathcal{M}$ by translating X to the space-time origin and parabolically rescaling by $1/r$. Here, $\mathcal{D}_\lambda(x, t) = (\lambda x, \lambda^2 t)$.

Definition 2.6 (ε-cylindrical)**.** We say that $\mathcal{M}$ is *ε-cylindrical around X at scale r*, if $\mathcal{M}_{X,r}$ is ε-close in $C^{\lfloor 1/\varepsilon \rfloor}$ in $B(0, 1/\varepsilon) \times [-1, -2]$ to the evolution of a round shrinking bubble-sheet cylinder with radius $r(t) = \sqrt{-2t}$ and axis through the origin.

Note that in [CHH18] and [CHHW19], being ε-cylindrical is defined by closeness to the evolution of a neck, rather than the evolution of a bubble-sheet. We hope that the benefits of the analogies coming from overloading the term outweigh the potential confusion. In any case, throughout the present paper ε-cylindrical is always meant in the sense of Definition 2.6.

Given any point $X = (x, t) \in \mathcal{M}$, we analyze the solution around X at the diadic scales $r_j = 2^j$, where $j \in \mathbb{Z}$.

Theorem 2.7. *For any small enough $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon) < \infty$ with the following significance. If $\mathcal{M}$ is an ancient flow asymptotic to a bubble-sheet, which is not the round shrinking bubble-sheet cylinder, then for every $X \in \mathcal{M}$ there exists an integer $J(X) \in \mathbb{Z}$ such that*

$$\mathcal{M} \textit{ is not } \varepsilon\textit{-cylindrical around } X \textit{ at scale } r_j \textit{ for all } j < J(X), \tag{28}$$

and

$$\mathcal{M} \textit{ is } \tfrac{\varepsilon}{2}\textit{-cylindrical around } X \textit{ at scale } r_j \textit{ for all } j \geqslant J(X) + N. \tag{29}$$

Proof. The proof, based on quantitative differentiation (c.f. [CHN13]), is identical to the one of [CHH18, Theorem 2.7]. □

We fix a small enough parameter $\varepsilon > 0$ quantifying the quality of the bubble-sheet for the rest of the paper.

Definition 2.8 (bubble-sheet scale)**.** The *bubble-sheet scale* of $X \in \mathcal{M}$ is defined by

$$Z(X) = 2^{J(X)}. \tag{30}$$

3. Foliations and barriers

The proof of Theorem 1.4 relies on the spectral analysis of the equation describing the evolution by mean curvature flow of a graph over the cylinder. Importantly, the flow is not an entire graph over a cylinder, but rather, a graph over a large portion of it. Thus, one needs to introduce some truncation, and control the error introduced by it. Moreover, one needs quantitative bounds on the graphical radius - the radius in which the flow is a small graph over the cylinder. In the neck-singularity setting, both of these goals are achieved in [ADS19] by using a foliation whose leaves are fixed points of the rescaled MCF. In this section, we construct analogues of that foliation which are suited for the analysis over the bubble-sheet cylinder $\mathbb{R}^2 \times S^1$.

We recall from Angenent-Daskalopoulos-Sesum that there exists some $L_0 > 1$ such that for every $a \geqslant L_0$ and $b > 0$ there are self-shrinkers

$$\begin{aligned} \Sigma_a &= \{\text{surface of revolution with profile } r = u_a(x_1), 0 \leqslant x_1 \leqslant a\}, \\ \tilde{\Sigma}_b &= \{\text{surface of revolution with profile } r = \tilde{u}_b(x_1), 0 \leqslant x_1 < \infty\}, \end{aligned} \tag{31}$$

as illustrated in [ADS19, Fig. 1], see also [KM14]. We will refer to these shrinkers as ADS-shrinkers and KM-shrinkers, respectively. Here, the parameter a captures where the concave functions u_a meet the x_1-axis, namely $u_a(a) = 0$, and the parameter b is the asymptotic slope of the convex functions $\tilde{u}_b$, namely $\lim_{x_1\to\infty} \tilde{u}_b'(x_1) = b$. A detailed description of these shrinkers can be found in [ADS19, Sec. 8]. We recall:

Lemma 3.1 (Lemmas 4.9 and 4.10 in [ADS19])**.** *There exists some $\delta > 0$ such that the self-shrinkers $\Sigma_a, \tilde{\Sigma}_b$ and the cylinder $\Sigma := \{x_2^2 + x_3^2 = 2\} \subset \mathbb{R}^3$ foliate the region*

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2^2 + x_3^2 \leqslant 2 + \delta \text{ and } x_1 \geqslant L_0\}. \tag{32}$$

Moreover, denoting by ν_{fol} the outward unit normal of this family, we have that

$$\operatorname{div}(e^{-x^2/4}\nu_{\text{fol}}) = 0, \tag{33}$$

i.e. the shrinker family forms a calibration for the Gaussian area.

We will now shift and rotate this foliation to construct a suitable foliation in $\mathbb{R}^4$:

Definition 3.2 (bubble-sheet foliation)**.** For every $a \geqslant L_0$, we denote by Γ_a the doubly-rotationally symmetric hypersurface in $\mathbb{R}^4$ given by

$$\Gamma_a = \{(r\cos\theta, r\sin\theta, x_3, x_4) \in \mathbb{R}^4 : \theta \in [0, 2\pi), (r-1, x_3, x_4) \in \Sigma_a\}. \tag{34}$$

Similarly, for each $b > 0$ we denote by $\tilde{\Gamma}_b$ the doubly-rotationally symmetric hypersurface in $\mathbb{R}^4$ given by

$$\tilde{\Gamma}_b = \{(r\cos\theta, r\sin\theta, x_3, x_4) \in \mathbb{R}^4 : \theta \in [0, 2\pi), (r-1, x_3, x_4) \in \tilde{\Sigma}_b\}. \tag{35}$$

Lemma 3.3 (Foliation lemma)**.** *There exist $\delta > 0$ and $L_1 > 2$ such that the hypersurfaces Γ_a, $\tilde{\Gamma}_b$, and the cylinder $\Gamma := \mathbb{R}^2 \times S^1(\sqrt{2})$ foliate the domain*

$$\Omega := \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_3^2 + x_4^2 \leqslant 2 + \delta \text{ and } x_1^2 + x_2^2 \geqslant L_1^2\right\}. \tag{36}$$

Moreover, denoting by ν_{fol} the outward pointing unit normal of this foliation, we have that

$$\mathrm{div}(\nu_{\mathrm{fol}} e^{-|x|^2/4}) \leqslant 0 \quad \text{inside the cylinder}, \tag{37}$$

and

$$\mathrm{div}(\nu_{\mathrm{fol}} e^{-|x|^2/4}) \geqslant 0 \quad \text{outside the cylinder}. \tag{38}$$

Proof. Let δ be as in Lemma 3.1, and set $L_1 = L_0 + 1$. The fact that Γ_a, $\tilde{\Gamma}_b$ and Γ foliate Ω follows from Lemma 3.1 and Definition 3.2.

Now, observe that for every element Γ_* in the foliation of Ω we have

$$\mathrm{div}(\nu_{\mathrm{fol}} e^{-|x|^2/4}) = \left(H_{\Gamma_*} - \tfrac{1}{2}\langle x, \nu_{\mathrm{fol}}\rangle\right) e^{-|x|^2/4}. \tag{39}$$

Hence, (37) is equivalent to the condition

$$H_{\Gamma_*} \leqslant \tfrac{1}{2}\langle x, \nu_{\mathrm{fol}}\rangle. \tag{40}$$

To show (40), note that by symmetry, it suffices to compute the curvatures H_{Γ_*} of Γ_* in the region $\{x_2 = 0,\ x_1 > 0\}$, where we can identify points and unit normals in Γ_* with the corresponding ones in Σ_*, by disregarding the x_2-component. The relation between the mean curvature of a surface $\Sigma_* \subset \mathbb{R}^3$ and its (unshifted) rotation $\Gamma_* \subset \mathbb{R}^4$ on points with $x_2 = 0$ and $x_1 > 0$ is given by

$$H_{\Gamma_*} = H_{\Sigma_*} + \frac{1}{x_1}\langle e_1, \nu\rangle, \tag{41}$$

where $e_1 = (1, 0, 0)$.

For the ADS-shrinkers, the concavity of u_a implies $\langle e_1, \nu_{\mathrm{fol}}\rangle \geqslant 0$, so using (41) and the shrinker equation we infer that

$$H_{\Gamma_a} = \frac{1}{2}\langle x - e_1, \nu_{\mathrm{fol}}\rangle + \frac{1}{x_1}\langle e_1, \nu_{\mathrm{fol}}\rangle \leqslant \frac{1}{2}\langle x, \nu_{\mathrm{fol}}\rangle, \tag{42}$$

as in $\Omega \cap \{x_2 = 0,\ x_1 > 0\}$, we have $x_1 \geqslant L_1 \geqslant 2$.

For the KM-shrinkers, the convexity of $\tilde{u}_b$ implies that $\langle e_1, \nu_{\mathrm{fol}}\rangle \leqslant 0$, so a similar calculation as in (42) gives

$$H_{\tilde{\Gamma}_b} \geqslant \frac{1}{2}\langle x, \nu_{\mathrm{fol}}\rangle. \tag{43}$$

This proves the lemma. □

Corollary 3.4 (Inner Barriers)**.** *The hypersurfaces Γ_a are inner barriers for rescaled MCF, in the following sense: Assume $\{K_\tau\}_{\tau\in[\tau_1,\tau_2]}$ are compact domains, the boundary of which evolve by rescaled MCF. Assume further that Γ_a is contained in K_τ for every $\tau \in [\tau_1, \tau_2]$, and that $\partial K_\tau \cap \Gamma_a = \emptyset$ for all $\tau < \tau_2$. Then*

$$\partial K_{\tau_2} \cap \Gamma_a \subseteq \partial \Gamma_a. \tag{44}$$

Proof. Lemma 3.3 implies that the vector $\vec{H} + \frac{x^\perp}{2}$ points outwards of Γ_a. The result now follows from the maximum principle. □

Remark 3.5 (Outer Barriers)**.** *Although this is not needed in the convex case, the hypersurfaces $\tilde{\Gamma}_b$ are evidently outer barriers for the rescaled MCF.*

4. Setting up the fine bubble-sheet analysis

Throughout this section, let $\mathcal{M}$ be a noncollapsed ancient flow in $\mathbb{R}^4$ whose tangent flow for $t \to -\infty$ is a bubble-sheet. Assume further that $\mathcal{M}$ is not self-similarly shrinking. Given any space-time point X_0, we consider the renormalized flow

$$\bar{M}^{X_0}_\tau = e^{\frac{\tau}{2}} \left(M_{t_0 - e^{-\tau}} - x_0 \right). \tag{45}$$

Then, $\bar{M}^{X_0}_\tau$ converges to the cylinder

$$\Gamma = \mathbb{R}^2 \times S^1(\sqrt{2}) \tag{46}$$

as $\tau \to -\infty$.

Since the renormalized MCF is invariant under rotations, the corresponding rotation vector fields appear as Jacobi fields in its linearization. Therefore, to obtain a useful geometric information from the spectral analysis in the case where the neutral mode is dominant, one needs to make sure that those rotation vector fields are not the dominant ones. There are two ways that have been successfully employed in doing that:

(1) using a neck-improvement theorem to show that the rotations are nondominant (c.f. [CHH18, Sec. 4.3]), or
(2) rotating the evolution in such a way that the graphs are orthogonal to all rotations (c.f. [BC19, Sec. 2]).

The difference between the two methods, in terms of the argument, is where those rotations are dealt with: in the latter approach, the labor lies in showing that the modes of the linearization dominate the evolution even after one rotates the surfaces. In the former approach, the analysis in the neutral mode case is harder.

In our current setting, we have found it easier to use the second alternative, as in [BC19].

After normalizing we can assume that $Z(X_0) \leqslant 1$, and we can find a universal function $\rho(\tau) > 0$ with

$$\lim_{\tau\to-\infty} \rho(\tau) = \infty, \quad \text{and} - \rho(\tau) \leqslant \rho'(\tau) \leqslant 0, \tag{47}$$

such that for every $S \in \mathrm{SO}(4)$ with $\sphericalangle(S(\Gamma), \Gamma) < \frac{1}{100}\rho(\tau)^{-3}$ the rotated surface $S(M_\tau^{X_0})$ is the graph of a function $u = u_S(\cdot, \tau)$ over $\Gamma \cap B_{2\rho(\tau)}(0)$, namely

$$\{x + u(x, \tau)\nu_\Sigma(x) : x \in \Gamma \cap B_{2\rho(\tau)}(0)\} \subset S(M_\tau^{X_0}), \tag{48}$$

where ν_Γ denotes the outward pointing unit normal to Γ, such that

$$\|u(\cdot, \tau)\|_{C^4(\Gamma\cap B_{2\rho(\tau)}(0))} \leqslant \rho(\tau)^{-2}. \tag{49}$$

We fix a nonnegative smooth cutoff function χ satisfying $\chi(s) = 1$ for $|s| \leqslant \frac{1}{2}$ and $\chi(s) = 0$ for $|s| \geqslant 1$. We consider the truncated function

$$\hat{u}(x, \tau) := u(x, \tau)\chi\left(\frac{r}{\rho(\tau)}\right), \tag{50}$$

where

$$r(x) := \sqrt{x_1^2 + x_2^2}. \tag{51}$$

Proposition 4.1 (Orthogonality)**.** *There exists a differentiable function $S^{X_0}(\tau) \in \mathrm{SO}(4)$, defined for τ sufficiently negative, such that, setting $u := u_{S^{X_0}(\tau)}$, we have*

$$\int_{\Gamma\cap B_{\rho(\tau)}(0)} \langle Ax, \nu_\Gamma\rangle \hat{u}(x, \tau)e^{-\frac{|x|^2}{4}} = 0, \tag{52}$$

for every $A \in \mathrm{o}(4)$. Moreover, the matrix $A(\tau) = S'(\tau)S(\tau)^{-1} \in \mathrm{o}(4)$ satisfies $A_{12}(\tau) = 0$ and $A_{34}(\tau) = 0$ for all $\tau \ll 0$.

Proof. Let $\mathrm{Gr}(2, \mathbb{R}^4)$ be the space of 2-dimensional planes through the origin in $\mathbb{R}^4$. The rotation group $\mathrm{O}(4)$ acts transitively on $\mathrm{Gr}(2, \mathbb{R}^4)$ with stabilizer $\mathrm{O}(2) \times \mathrm{O}(2)$, and hence the Grassmannian can be expressed as homogeneous space

$$\mathrm{Gr}(2, \mathbb{R}^4) = \frac{\mathrm{O}(4)}{\mathrm{O}(2) \times \mathrm{O}(2)}. \tag{53}$$

In particular, $\dim \mathrm{Gr}(2, \mathbb{R}^4) = 6 - 2 = 4$. Let us select an explicit choice of fibration

$$p : \mathrm{O}(4) \to \mathrm{Gr}(2, \mathbb{R}^4), \tag{54}$$

by declaring that p maps each rotation matrix to the span of its first two column vectors.

Now, there is a neighbourhood U_0 of the identity matrix $I \in \mathrm{O}(4)$ such that for every $R \in U_0$ and every $\tau \ll 0$, we can write $R(\bar{M}_\tau)$ as a graph of a function $u_R(x, \tau)$ over $\Gamma \cap B_{2\rho(\tau)}(0)$. Observe that the expression

$$\int_{\Gamma\cap B_{\rho(\tau)}(0)} e^{-|x|^2/4} u_R^2(x, \tau)\chi\left(\frac{r}{\rho(\tau)}\right), \tag{55}$$

as well as the relation

$$\int_{\Gamma\cap B_{\rho(\tau)}(0)} e^{-|x|^2/4}\langle Ax,\nu_\Gamma\rangle u_R(x,\tau)\chi\left(\frac{r}{\rho(\tau)}\right) = 0 \quad \forall A\in o(4), \tag{56}$$

is constant along the fibers of p. Set $V_0 := p(U_0)$, and observe that this is a neighborhood of $[x_1x_2]\in \mathrm{Gr}(2,\mathbb{R}^4)$.

Claim 4.2. Perhaps after decreasing U_0, for every every $\tau \ll 0$ there exists a unique $\pi(\tau)\in V_0$ such that

$$\int_{\Gamma\cap B_{\rho(\tau)}(0)} e^{-|x|^2/4}\langle Ax,\nu_\Gamma\rangle u_{\tilde\pi(\tau)}(x,\tau)\chi\left(\frac{r}{\rho(\tau)}\right) = 0 \quad \forall A\in \mathrm{o}(4), \tag{57}$$

where $\tilde\pi(\tau)$ denotes some lift of $\pi(\tau)$ to $\mathrm{O}(4)$. Moreover, the function $\tau\mapsto\pi(\tau)$ is smooth.

Proof of the claim. We will use the quantitative version of the implicit function theorem.

In order to first construct the required approximate solution, we consider the functional

$$H(\tau,\pi) = \frac{1}{2}\int_{\Gamma\cap B_{\rho(\tau)}(0)} e^{-|x|^2/4}u_\pi^2(x,\tau)\chi\left(\frac{r}{\rho(\tau)}\right). \tag{58}$$

Since M_τ is a small graph over the cylinder with axis given by the 2-plane $[x_1x_2]\in \mathrm{Gr}(2,\mathbb{R}^4)$, for every $\tau\ll 0$ there is a minimizer $\pi(\tau)$ for H in a small neighbourhood of $[x_1x_2]$, and we have $H(\tau,\pi(\tau))\leqslant C/\rho(\tau)^4$. Fix some $\tau\ll 0$ and assume the minimum is obtained at $[x_1x_2]$ (which we can do after rotating M_τ).
Now, if $R(\eta)\in \mathrm{SO}(4)$ is a one-parameter family of rotations with $R(0)=I$ and $R'(0)=A\in \mathrm{o}(4)$, then an elementary computation shows that

$$\frac{d}{d\eta}|_{\eta=0}u_{R(\eta)}(x,\tau) = \langle Ax,\nu_\Gamma\rangle + O\left(\frac{|x|}{\rho(\tau)^2}\right). \tag{59}$$

Using this, we obtain that the Euler-Lagrange equation for H reads

$$\begin{aligned}\nabla_A H &= \int_{\Gamma\cap B_{\rho(\tau)}(0)} e^{-|x|^2/4}u_\pi(x,\tau)\langle Ax,\nu_\Gamma\rangle\chi\left(\frac{r}{\rho(\tau)}\right) + O\left(\frac{1}{\rho(\tau)^4}\right)\\ &= 0 \quad \forall A\in o(4).\end{aligned} \tag{60}$$

Thus, our minimizer $\pi(\tau)$ is an approximate solution of (57).

Now, to show existence and uniqueness of solutions to (57), as well as smooth dependence, let $A^1,\dots A^4$ be the standard basis of

$$\mathcal{A} := \{A\in so(4)\mid A_{12}=A_{34}=0\}, \tag{61}$$

and define a map $F:(-\infty,\tau_0)\times V_0\to\mathbb{R}^4$ by

$$F(\tau,\pi)_i = \int_{\Gamma\cap B_{\rho(\tau)}(0)} e^{-|x|^2/4}\langle A^ix,\nu_\Gamma\rangle u_\pi(x,\tau)\chi\left(\frac{r}{\rho(\tau)}\right) \tag{62}$$

for $i = 1, \ldots, 4$. Then, computing similarly as above we obtain

$$\nabla_{A^i} F_j = \int_{\Gamma \cap B_{\rho(\tau)}(0)} e^{-|x|^2/4} \langle A^i x, \nu_\Gamma \rangle \langle A^j x, \nu_\Gamma \rangle \chi\left(\frac{r}{\rho(\tau)}\right) + O\left(\frac{1}{\rho(\tau)^2}\right). \tag{63}$$

Since the only anti-symmetric matrix in $A \in \mathcal{A}$ such that $\langle Ax, \nu_\Gamma \rangle \equiv 0$ on Γ is 0, we see that the form $\nabla_{A^i} F_j$ is positive definite for $\tau \ll 0$, and so is in particular invertible.

As, by symmetry, our calculation did not rely on being in the specific point $\pi(\tau) = [x_1 x_2]$, equation (63) holds in an $\rho(\tau)^{-3}/100$ neighborhood of $[x_1 x_2]$ with uniform constants. Thus by the quantitative version of the inverse function theorem, in light of (60) and (63), there is a neighborhood of our fixed time τ such that the function $\pi(\cdot)$ can be chosen in it to satisfy (57). Finally, since by the above F is uniformly bounded in spatial C^1, the size of that neighbourhood can be taken to be independent of τ. This finishes the proof of the claim. □

Now, we can take a lift $S(\tau)$ of $\pi(\tau)$ to SO(4). Moreover, by post-composing with two SO(2) rotations, of the first two variables, and of the last two variables, we can further arrange that $A = S'S^{-1}$ satisfies $A_{12} = A_{34} = 0$. This finishes the proof of the proposition. □

We now set

$$\tilde{M}_\tau^{X_0} = S^{X_0}(\tau) \bar{M}_\tau^{X_0}, \tag{64}$$

where $S^{X_0}(\tau) \in \mathrm{SO}(4)$ if from Proposition 4.1 (Orthogonality), and set

$$u := u_{S^{X_0}(\tau)}, \tag{65}$$

so u describes $\tilde{M}_\tau^{X_0}$ as a graph over the cylinder Γ. Recall that we defined

$$\hat{u}(x,\tau) = u(x,\tau)\chi\left(\frac{r}{\rho(\tau)}\right). \tag{66}$$

Our next task it to show that, despite of the truncation and the rotation, the function $\hat{u}$ evolves by the linearization of the rescaled MCF equation over the cylinder, up to a small error. Set

$$e_r := \nabla r = \frac{(x_1, x_2, 0, 0)}{r}. \tag{67}$$

Proposition 4.3 (Gaussian area)**.** *Let Δ_τ be the region bounded by $\tilde{M}_\tau$ and Γ. For all $L \in [L_1, \rho(\tau)]$ and $\tau \ll 0$ we have the Gaussian area estimate*

$$\int_{\tilde{M}_\tau \cap \{r \geqslant L\}} e^{-\frac{|x|^2}{4}} - \int_{\Gamma \cap \{r \geqslant L\}} e^{-\frac{|x|^2}{4}} \geqslant -\int_{\Delta_\tau \cap \{r = L\}} e^{-\frac{|x|^2}{4}} |\langle e_r, \nu_{\mathrm{fol}} \rangle|. \tag{68}$$

Proof. Let $\tilde{M}_\tau^+$ (respectively $\tilde{M}_\tau^-$) be the part of $\tilde{M}_\tau$ that lies outside (respectively inside) the cylinder. Let $\Delta_\tau^\pm$ be the corresponding parts of Δ_τ, and let $\Gamma^\pm = \Delta_\tau^\pm \cap \Gamma$.

Considering the foliation of Ω from Lemma 3.3, since $\mathrm{div}(e^{-\frac{|x|^2}{4}}\nu_{\mathrm{fol}}) \geqslant 0$ in Δ_τ^+, the divergence theorem yields that for every $R > L$ we have

$$\int_{\tilde{M}_\tau^+\cap\{L\leqslant r\leqslant R\}} e^{-\frac{|x|^2}{4}} \langle \nu, \nu_{\mathrm{fol}}\rangle - \int_{\Gamma^+\cap\{L\leqslant r\leqslant R\}} e^{-\frac{|x|^2}{4}} \geqslant -\int_{\Delta_\tau^+\cap\{r=L\}} e^{-\frac{|x|^2}{4}} |\langle e_r, \nu_{\mathrm{fol}}\rangle| - \int_{\Delta_\tau^+\cap\{r=R\}} e^{-\frac{|x|^2}{4}} |\langle e_r, \nu_{\mathrm{fol}}\rangle|. \tag{69}$$

Similarly, since $\mathrm{div}(e^{-\frac{|x|^2}{4}}\nu_{\mathrm{fol}}) \leqslant 0$ in Δ_τ^-, the divergence theorem yields

$$\int_{\Gamma^-\cap\{L\leqslant r\leqslant R\}} e^{-\frac{|x|^2}{4}} - \int_{\tilde{M}_\tau^-\cap\{L\leqslant r\leqslant R\}} e^{-\frac{|x|^2}{4}} \langle \nu, \nu_{\mathrm{fol}}\rangle \leqslant \int_{\Delta_\tau^-\cap\{r=L\}} e^{-\frac{|x|^2}{4}} |\langle e_r, \nu_{\mathrm{fol}}\rangle| + \int_{\Delta_\tau^-\cap\{r=R\}} e^{-\frac{|x|^2}{4}} |\langle e_r, \nu_{\mathrm{fol}}\rangle|. \tag{70}$$

Putting these together, we get

$$\int_{\tilde{M}_\tau\cap\{L\leqslant r\leqslant R\}} e^{-\frac{|x|^2}{4}} \langle \nu, \nu_{\mathrm{fol}}\rangle - \int_{\Gamma\cap\{L\leqslant r\leqslant R\}} e^{-\frac{|x|^2}{4}} \geqslant -\int_{\Delta_\tau\cap\{r=L\}} e^{-\frac{|x|^2}{4}} |\langle e_r, \nu_{\mathrm{fol}}\rangle| - \int_{\Delta_\tau\cap\{r=R\}} e^{-\frac{|x|^2}{4}} |\langle e_r, \nu_{\mathrm{fol}}\rangle|. \tag{71}$$

Using $|\langle \nu, \nu_{\mathrm{fol}}\rangle| \leqslant 1$ and $|\Delta_\tau \cap \{r = R\}| \leqslant CR^3$ and passing $R \to \infty$ we conclude that

$$\int_{\tilde{M}_\tau\cap\{r\geqslant L\}} e^{-\frac{|x|^2}{4}} - \int_{\Gamma\cap\{r\geqslant L\}} e^{-\frac{|x|^2}{4}} \geqslant -\int_{\Delta_\tau\cap\{r=L\}} e^{-\frac{|x|^2}{4}} |\langle e_r, \nu_{\mathrm{fol}}\rangle|. \tag{72}$$

This proves the proposition. □

Next, we have an inverse Poincare inequality:

Proposition 4.4 (Inverse Poincare inequality)**.** *The graph function u satisfies the integral estimates*

$$\int_{\Gamma\cap\{|r|\leqslant L\}} e^{-\frac{|x|^2}{4}} |\nabla u(x,\tau)|^2 \leqslant C \int_{\Gamma\cap\{|r|\leqslant \frac{L}{2}\}} e^{-\frac{|x|^2}{4}} u(x,\tau)^2 \tag{73}$$

and

$$\int_{\Gamma\cap\{\frac{L}{2}\leqslant|r|\leqslant L\}} e^{-\frac{|x|^2}{4}} u(x,\tau)^2 \leqslant \frac{C}{L^2} \int_{\Gamma\cap\{|r|\leqslant \frac{L}{2}\}} e^{-\frac{|x|^2}{4}} u(x,\tau)^2 \tag{74}$$

for all $L \in [L_1, \rho(\tau)]$ and $\tau \ll 0$, where $C < \infty$ is a numerical constant.

Proof. The proof is quite similar to the one of Proposition 2.3 in [BC19]. Since $|\langle e_r, \nu_{\mathrm{fol}}\rangle| \leqslant CL^{-1}|x_3^2 + x_4^2 - 2|$ by Lemma 4.11 in [ADS19], we infer that

$$\int_{\Delta_\tau\cap\{r=L\}} e^{-\frac{|x|^2}{4}} |\langle e_r, \nu_{\mathrm{fol}}\rangle| \leqslant CL^{-1} \int_{\Gamma\cap\{r=L\}} e^{-\frac{|x|^2}{4}} u^2. \tag{75}$$

Thus, Proposition 4.3 (Gaussian area) implies

$$\int_{\tilde{M}_\tau \cap \{r \geqslant L\}} e^{-\frac{|x|^2}{4}} - \int_{\Gamma \cap \{r \geqslant L\}} e^{-\frac{|x|^2}{4}} \geqslant -CL^{-1} \int_{\Gamma \cap \{r = L\}} e^{-\frac{|x|^2}{4}} u^2 . \tag{76}$$

On the other hand, we have

$$\begin{aligned} \int_{\tilde{M}_\tau \cap \{r \leqslant L\}} e^{-\frac{|x|^2}{4}} - \int_{\Gamma \cap \{r \leqslant L\}} e^{-\frac{|x|^2}{4}} \\ \geqslant \int_{\{r \leqslant L\}} \int_0^{2\pi} e^{-\frac{r^2}{4}} \Big[- Cu^2 + \frac{1}{C} |\nabla^\Gamma u|^2 \Big] \, d\theta \, dA, \end{aligned} \tag{77}$$

where $C < \infty$ is a numerical constant. Hence, we can do the same computation as in the proof of Proposition 2.3 in [BC19] to obtain the desired result. □

Recall that $\tilde{M}_\tau$ is expressed as graph of a function $u(x,\tau)$ over $\Gamma \cap B_{2\rho(\tau)}(0)$ satisfying the estimate (49). Using that $\bar{M}_\tau = S(\tau)^{-1} \tilde{M}_\tau$ moves by rescaled mean curvature flow one obtains:

Lemma 4.5 (evolution of graph function)**.** *The function $u(x,\tau)$ satisfies*

$$\partial_\tau u = \mathcal{L} u + E + \langle A(\tau) x, \nu_\Gamma \rangle, \tag{78}$$

where $A = S' S^{-1}$ and $\mathcal{L}$ is the linear operator on Γ defined by

$$\mathcal{L} f = \Delta f - \frac{1}{2} \langle x^{\tan}, \nabla f \rangle + f, \tag{79}$$

and where the error term can be estimated by

$$|E| \leqslant C \rho^{-1}(\tau) (|u| + |\nabla u| + |A(\tau)|) \tag{80}$$

for $\tau \ll 0$.

Proof. The proof is identical to the proof of [BC19, Lemma 2.4]. □

Denote by $\mathcal{H}$ the Hilbert space of all functions f on Γ such that

$$\|f\|_{\mathcal{H}}^2 = \int_\Gamma \frac{1}{(4\pi)^{3/2}} e^{-\frac{|x|^2}{4}} f^2 < \infty. \tag{81}$$

Proposition 4.6 (evolution of truncated graph function)**.** *The truncated function $\hat{u}(x,\tau) = u(x,\tau) \chi\big(\frac{r}{\rho(\tau)}\big)$ satisfies*

$$\partial_\tau \hat{u} = \mathcal{L} \hat{u} + \hat{E} + \langle A(\tau) x, \nu_\Gamma \rangle \chi \left(\frac{r}{\rho(\tau)} \right), \tag{82}$$

where

$$\|\hat{E}\|_{\mathcal{H}} \leqslant C \rho^{-1} \big(\|\hat{u}\|_{\mathcal{H}} + |A(\tau)| \big) \tag{83}$$

for $\tau \ll 0$.

Proof. We compute

$$\partial_\tau \hat{u} = \mathcal{L}\hat{u} + \hat{E} + \langle A(\tau)x, \nu_\Gamma\rangle \chi\Big(\frac{r}{\rho(\tau)}\Big), \tag{84}$$

where

$$\begin{aligned}\hat{E} = E\chi\Big(\frac{r}{\rho(\tau)}\Big) - \frac{2}{\rho(\tau)}\frac{\partial u}{\partial r}\chi'\Big(\frac{r}{\rho(\tau)}\Big) - \frac{1}{\rho(\tau)^2}u\chi''\Big(\frac{r}{\rho(\tau)}\Big)\\ + \frac{r}{2\rho(\tau)}u\chi'\Big(\frac{r}{\rho(\tau)}\Big) - \frac{r\rho'(\tau)}{\rho(\tau)^2}u\chi'\Big(\frac{r}{\rho(\tau)}\Big).\end{aligned} \tag{85}$$

Using Lemma 4.5 (evolution of graph function), we deduce that

$$|\hat{E}| \leqslant C\rho(\tau)^{-1}(|u| + |\nabla^\Gamma u| + |A(\tau)|)$$

for $r \leqslant \frac{\rho(\tau)}{2}$. Moreover, since $|\rho'(\tau)| \leqslant \rho(\tau)$ and $\rho(\tau) \to \infty$, we obtain

$$|\hat{E}| \leqslant C|u| + C\rho^{-1}(|\nabla^\Gamma u| + |A(\tau)|)$$

for $\frac{\rho(\tau)}{2} \leqslant r \leqslant \rho(\tau)$. Thus, we can obtain the desired result as in the proof of [BC19, Lemma 2.5]. □

Lemma 4.7 (estimate for rotation function)**.** *The rotation can be estimated by*

$$|A(\tau)| \leqslant C\rho^{-1}\|u\|_{\mathcal{H}}. \tag{86}$$

In particular, we have

$$\|\hat{u}_\tau - \mathcal{L}\hat{u}\|_{\mathcal{H}} \leqslant C\rho^{-1}\|\hat{u}\|_{\mathcal{H}}. \tag{87}$$

Proof. The proof is similar to the one of [BC19, Lemma 2.6]. The conditions $A_{12}(\tau) = A_{34}(\tau) = 0$ imply

$$|A(\tau)|^2 \leqslant C\int_\Gamma e^{-\frac{|x|^2}{4}}\langle A(\tau)x, \nu_\Gamma\rangle^2 \chi\Big(\frac{r}{\rho(\tau)}\Big). \tag{88}$$

Indeed, since non zero anti-symmetric matrices A with $A_{12} = A_{34} = 0$ are the velocity fields of rotations which change the axis of the cylinder, the bilinear-form

$$(A, B) := \int_\Gamma e^{-\frac{|x|^2}{4}}\langle Ax, \nu_\Gamma\rangle\langle Bx, \nu_\Gamma\rangle\chi\Big(\frac{r}{\rho(\tau)}\Big), \tag{89}$$

is uniformly positive definite on such matrices.
Proposition 4.1 (orthogonality) implies that $\hat{u}$, hence also $\partial_\tau\hat{u}$, is orthogonal to all $\langle Ax, \nu\rangle$ for each anti-symmetric A. Also $L\hat{u}$ is orthogonal to $\langle Ax, \nu\rangle$, as the latter is in the kernel of L. Thus, $\langle Ax, \nu\rangle$ is orthogonal to $\partial_\tau\hat{u} - L\hat{u} = \hat{E} + \langle A(\tau)x, \nu_\Gamma\rangle\chi\big(\frac{r}{\rho(\tau)}\big)$. From this and (88), we get

$$\begin{aligned}|A(\tau)|^2 &\leqslant C\int_\Gamma e^{-\frac{|x|^2}{4}}\,|\hat{E}|\,|\langle A(\tau)x, \nu_\Gamma\rangle|\chi\Big(\frac{r}{\rho(\tau)}\Big)\\ &\leqslant C\,\|\hat{E}\|_{\mathcal{H}}\,|A(\tau)|\\ &\leqslant C\rho^{-1}\,\|\hat{u}\|_{\mathcal{H}}\,|A(\tau)| + C\rho^{-1}\,|A(\tau)|^2,\end{aligned}$$

where in the last step we have used Proposition 4.6. Consequently,

$$|A(\tau)| \leqslant C\rho^{-1} \|\hat{u}\|_{\mathcal{H}}. \tag{90}$$

Using Proposition 4.6 once more, we get (87) as well. □

The operator $\mathcal{L}$ is explicitly given by

$$\mathcal{L} = \frac{\partial^2}{\partial x_1^2} f + \frac{\partial^2}{\partial x_2^2} f + \frac{1}{2} \frac{\partial^2}{\partial \theta^2} f - \frac{1}{2} x_1 \frac{\partial}{\partial x_1} f - \frac{1}{2} x_2 \frac{\partial}{\partial x_2} f + f. \tag{91}$$

Analysing the spectrum of $\mathcal{L}$, the Hilbert space $\mathcal{H}$ from (81) can be decomposed as

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-, \tag{92}$$

where

$$\mathcal{H}_+ = \operatorname{span}\{1, \cos\theta, \sin\theta, x_1, x_2\}, \tag{93}$$

$$\mathcal{H}_0 = \operatorname{span}\{x_1 \cos\theta, x_1 \sin\theta, x_2 \cos\theta, x_2 \sin\theta, x_1^2 - 2, x_2^2 - 2, x_1 x_2\}. \tag{94}$$

We have

$$\begin{aligned} \langle \mathcal{L} f, f \rangle_{\mathcal{H}} &\geqslant \frac{1}{2} \|f\|_{\mathcal{H}}^2 && \text{for } f \in \mathcal{H}_+, \\ \langle \mathcal{L} f, f \rangle_{\mathcal{H}} &= 0 && \text{for } f \in \mathcal{H}_0, \\ \langle \mathcal{L} f, f \rangle_{\mathcal{H}} &\leqslant -\frac{1}{2} \|f\|_{\mathcal{H}}^2 && \text{for } f \in \mathcal{H}_-. \end{aligned} \tag{95}$$

Consider the functions

$$\begin{aligned} U_+(\tau) &:= \|P_+ \hat{u}(\cdot, \tau)\|_{\mathcal{H}}^2, \\ U_0(\tau) &:= \|P_0 \hat{u}(\cdot, \tau)\|_{\mathcal{H}}^2, \\ U_-(\tau) &:= \|P_- \hat{u}(\cdot, \tau)\|_{\mathcal{H}}^2, \end{aligned} \tag{96}$$

where P_+, P_0, P_- denote the orthogonal projections to $\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-$, respectively.

Theorem 4.8 (Merle-Zaag alternative)**.** *For $\tau \to -\infty$ either the neutral mode is dominant, i.e.*

$$U_- + U_+ = o(U_0), \tag{97}$$

or the unstable mode is dominant, i.e.

$$U_- + U_0 \leqslant C\rho^{-1} U_+. \tag{98}$$

Proof. Using Proposition 4.6 (evolution of truncated graph function) and Lemma 4.7 (estimate for rotation function) we obtain

$$\begin{aligned} \frac{d}{d\tau} U_+(\tau) &\geqslant U_+(\tau) - C\rho^{-1} \left(U_+(\tau) + U_0(\tau) + U_-(\tau)\right), \\ \left|\frac{d}{d\tau} U_0(\tau)\right| &\leqslant C\rho^{-1} \left(U_+(\tau) + U_0(\tau) + U_-(\tau)\right), \\ \frac{d}{d\tau} U_-(\tau) &\leqslant -U_-(\tau) + C\rho^{-1} \left(U_+(\tau) + U_0(\tau) + U_-(\tau)\right). \end{aligned} \tag{99}$$

Hence, the Merle-Zaag ODE lemma (Lemma B.1) implies the assertion. □

5. Bubble-sheet analysis in the neutral mode

In this section, we prove Theorem 1.4 in the case where the neutral mode is dominant. Namely, throughout this section we assume that our truncated graph function $\hat{u}(\cdot,\tau)$ satisfies

$$U_- + U_+ = o(U_0). \tag{100}$$

The following lemma gives a rough bound, showing that for $\tau \to -\infty$ the function U_0 decays slower than any exponential.

Lemma 5.1 (rough decay estimate)**.** *For any $\delta > 0$ we have*

$$U_0(\tau) \geqslant e^{\delta\tau} \tag{101}$$

for sufficiently large $-\tau$.

Proof. Given any $\delta > 0$, the inequality (99) together with the assumption $U_- + U_+ = o(U_0)$ implies that

$$\left|\frac{d}{d\tau}U_0(\tau)\right| \leqslant \tfrac{1}{2}\delta U_0 \tag{102}$$

for sufficiently large $-\tau$. Rewriting this as

$$\left|\frac{d}{d\tau}\log U_0(\tau)\right| \leqslant \tfrac{1}{2}\delta, \tag{103}$$

integration gives $\log U_0(\tau) \geqslant -C + \frac{1}{2}\delta\tau$. This yields the desired result. □

Proposition 5.2. *Every sequence $\{\tau_i\}$ converging to $-\infty$ has a subsequence $\{\tau_{i_m}\}$ such that*

$$\lim_{\tau_{i_m}\to-\infty} \frac{\hat{u}(\cdot,\tau_{i_m})}{\|\hat{u}(\cdot,\tau_{i_m})\|_{\mathcal{H}}} = q_{11}(x_1^2-2) + q_{22}(x_2^2-2) + 2q_{12}x_1x_2 \tag{104}$$

in $\mathcal{H}$-norm, where $\{q_{ij}\}$ is a nontrivial semi-negative definite 2×2-matrix.

Proof. By Proposition 4.1 (orthogonality), we have

$$P_0\hat{u} \in \operatorname{span}\{x_1^2-2, x_2^2-2, x_1x_2\} \subset \mathcal{H}_0. \tag{105}$$

Therefore, every sequence $\tau_i \to -\infty$ has a subsequence $\{\tau_{i_m}\}$ such that

$$\lim_{\tau_{i_m}\to-\infty} \frac{\hat{u}(\cdot,\tau_{i_m})}{\|\hat{u}(\cdot,\tau_{i_m})\|_{\mathcal{H}}} = Q(x_1,x_2), \tag{106}$$

with respect to the $\mathcal{H}$-norm, where

$$Q(x_1,x_2) = q_{11}(x_1^2-2) + q_{22}(x_2^2-2) + 2q_{12}x_1x_2, \tag{107}$$

for some nontrivial matrix $Q = \{q_{ij}\}$. After an orthogonal change of coordinates in the x_1x_2-plane we can assume that $q_{12} = 0$. Let us show that $q_{11} \leqslant 0$ (the same argument yields $q_{22} \leqslant 0$).

We denote by $\tilde{K}_\tau$ the region enclosed by $\tilde{M}_\tau$ and denote by $\mathcal{A}(x_1', x_2', \tau)$ the area of the cross section $\tilde{K}_\tau \cap \{(x_1, x_2) = (x_1', x_2')\}$. Explicitly,

$$\begin{aligned}\mathcal{A}(x_1, x_2, \tau) &= \frac{1}{2}\int_0^{2\pi} \left(\sqrt{2} + u(\theta, x_1, x_2, \tau)\right)^2 d\theta \\ &= 2\pi + \sqrt{2}\int_0^{2\pi} u(\theta, x_1, x_2, \tau)d\theta + \tfrac{1}{2}\int_0^{2\pi} u(\theta, x_1, x_2, \tau)^2 d\theta \,. \end{aligned} \tag{108}$$

By Brunn's concavity principle the function

$$(x_1, x_2) \mapsto \sqrt{\mathcal{A}(x_1, x_2, \tau)} \tag{109}$$

is concave. In particular, we have

$$\sqrt{\mathcal{A}(x_1, x_2, \tau)} \geqslant \tfrac{1}{2}\sqrt{\mathcal{A}(x_1 - 2, x_2, \tau)} + \tfrac{1}{2}\sqrt{\mathcal{A}(x_1 + 2, x_2, \tau)}. \tag{110}$$

This implies

$$3\int_{-1}^{1}\int_{-1}^{1} \sqrt{\mathcal{A}}\, dx_2 dx_1 \geqslant \int_{-3}^{3}\int_{-1}^{1} \sqrt{\mathcal{A}}\, dx_2 dx_1. \tag{111}$$

Hence, for sufficiently large $-\tau_{i_m}$ combining (106), (108) and (111) yields

$$\begin{aligned}(3 - o(1))\|\hat{u}\|_{\mathcal{H}} &\int_{-1}^{1}\int_{-1}^{1} Q\, dx_2 dx_1 \\ &\geqslant (1 + o(1))\|\hat{u}\|_{\mathcal{H}} \int_{-3}^{3}\int_{-1}^{1} Q\, dx_2 dx_1 - O(\|\hat{u}\|_{\mathcal{H}}^2). \end{aligned} \tag{112}$$

Since $\|\hat{u}\|_{\mathcal{H}} > 0$ and $\|\hat{u}\|_{\mathcal{H}} \to 0$, taking the limit as $m \to \infty$, we obtain

$$3\int_{-1}^{1}\int_{-1}^{1} Q\, dx_2 dx_1 \geqslant \int_{-3}^{3}\int_{-1}^{1} Q\, dx_2 dx_1. \tag{113}$$

This implies

$$3q_{11}\int_{-1}^{1}\int_{-1}^{1} (x_1^2 - 2)\, dx_2 dx_1 \geqslant q_{11}\int_{-3}^{3}\int_{-1}^{1} (x_1^2 - 2)\, dx_2 dx_1. \tag{114}$$

Since the integral of the left hand side is negative, while the integral on the right hand side is positive, we conclude that $q_{11} \leqslant 0$. This proves the proposition. □

Recall that in this section $M_t = \partial K_t$ denotes an ancient noncollapsed flow in $\mathbb{R}^4$ with dominant neutral mode, i.e. (100) holds. Now, by the reduction from Section 2, we can assume that its blowdown $\check{K} \equiv \check{K}_{t_0}$ is a convex cone that does not contain any line. Furthermore, rotating coordinates, we can arrange that $\check{K}$ contains the positive x_1-axis, namely

$$\{\lambda e_1 \mid \lambda \geqslant 0\} \subseteq \check{K}. \tag{115}$$

Theorem 5.3 (blowdown in neutral mode)**.** *If $M_t = \partial K_t$ is as above, then its blowdown is a halfline, namely*

$$\check{K} = \{\lambda e_1 \mid \lambda \geqslant 0\}. \tag{116}$$

Moreover,

$$\lim_{\tau\to-\infty} \frac{\hat{u}(\cdot,\tau)}{\|\hat{u}(\cdot,\tau)\|_{\mathcal{H}}} = -c(x_2^2-2) \tag{117}$$

in $\mathcal{H}$-norm, where $c = ||x_2^2-2||_{\mathcal{H}}$.

Proof. By Proposition 5.2, we know that

$$\lim_{\tau_{i_m}\to-\infty} \frac{\hat{u}(\cdot,\tau_{i_m})}{\|\hat{u}(\cdot,\tau_{i_m})\|_{\mathcal{H}}} = q_{11}(x_1^2-2)+q_{22}(x_2^2-2)+2q_{12}x_1x_2 \tag{118}$$

in $\mathcal{H}$-norm, where $Q = \{q_{ij}\}$ is a nontrivial semi-negative definite 2×2-matrix. We will show that

$$\check{K} \subseteq \ker Q. \tag{119}$$

To this end, observe that if v is a unit vector in the x_1x_2-plane, denoting by w the unit vector that is obtained from v by a $\pi/2$-rotation, then

$$\int_0^1\int_0^1 Q(rv+sw)drds > 0. \tag{120}$$

On the other hand, if $v \notin \ker Q$, we see that for sufficiently large $d = d(\measuredangle(v,\ker Q))$, we have

$$\int_0^1\int_d^{d+1} Q(rv+sw)drds < 0. \tag{121}$$

Let $\tilde{K}_\tau$ be, as before, the region enclosed by $\tilde{M}_\tau$. Defining $\mathcal{A}$ as in (108), similarly as in the previous proof we have

$$\int_0^1\int_a^b \frac{\sqrt{A(rv+sw,\tau_m)}-\sqrt{2\pi}}{||\hat{u}||_{\mathcal{H}}}drds \to \frac{1}{\sqrt{4\pi}}\int_0^1\int_a^b Q(rv+sw)drds \tag{122}$$

as $m\to\infty$. Combining (120), (121) and (122) we see that for every m sufficiently large, there exists $r_m, s_m \in [0,1]$ such that

$$\mathcal{A}(r_mv+s_mw,\tau_m) > \mathcal{A}((r_m+d)v+s_mw,\tau_m). \tag{123}$$

Now, suppose towards a contradiction there is some $\omega \in \check{K}\setminus\ker Q$. Since $S(\tau)\to I$ as $\tau\to-\infty$, for all $-\tau$ sufficiently large we have

$$\measuredangle(P_{12}(S(\tau)\omega),\ker Q) > \tfrac{1}{2}\measuredangle(\omega,\ker Q), \tag{124}$$

where P_{12} denotes the projection to the span$\{e_1,e_2\}$. Set

$$v_m := \frac{P_{12}(S(\tau_m)\omega)}{|P_{12}(S(\tau_m)\omega)|}. \tag{125}$$

Taking $v = v_m$ in the previous discussion (and letting w_m its $\pi/2$-rotation), as $r_mv_m+s_mw_m \in \tilde{K}_{\tau_m}$, we see that

$$r_mv_m+s_mw_m+\lambda S(\tau_m)\omega \in \tilde{K}_{\tau_m} \quad \text{for every } \lambda\in[0,\infty). \tag{126}$$

On the other hand, thanks to Brunn's concavity principle, the function

$$r \mapsto \sqrt{\mathcal{A}(rv_m + s_m w_m, \tau)} \tag{127}$$

is concave, for as long as it does not vanish. Together with (123) this implies that for all m sufficiently large, the area of the cross sections is decreasing for $r > r_m$, and vanish at some finite r_*. This contradicts (126), as the ray would have nowhere to go. This proves (119).

Using the inclusion (119), since $\ker Q$ is 1-dimensional, we infer that $\check{K}$ is 1-dimensional. It follows that

$$\check{K} = \{\lambda e_1 \mid \lambda \geqslant 0\}, \tag{128}$$

and

$$\ker Q = \{x_1\text{-axis}\}. \tag{129}$$

Finally, since a normalized semi-negative 2×2-matrix is uniquely characterized by its 1-dimensional kernel, we see that subsequential convergence in (118) in fact entails full convergence, and

$$\lim_{\tau \to -\infty} \frac{\hat{u}(\cdot, \tau)}{\|\hat{u}(\cdot, \tau)\|_{\mathcal{H}}} = -c(x_2^2 - 2), \tag{130}$$

where $c = ||x_2^2 - 2||_{\mathcal{H}}$. This proves the theorem. □

Corollary 5.4 (diameter of level sets)**.** *If $M_t = \partial K_t$ is as above, then given any X and $\delta > 0$ we have*

$$\bar{M}_\tau^X \cap \{x_1 = 0\} \subset B_{e^{-\delta\tau}}(0)$$

for sufficiently large $-\tau$.

Proof. By Theorem 5.3 we have

$$\lim_{\tau \to -\infty} \frac{\hat{u}(\cdot, \tau)}{\|\hat{u}(\cdot, \tau)\|_{\mathcal{H}}} = -\frac{(x_2^2 - 2)}{||x_2^2 - 2||_{\mathcal{H}}}. \tag{131}$$

Hence, arguing similarly as above, for sufficiently large $-\tau$ we obtain

$$\int_{-1}^{1}\int_{0}^{1} \sqrt{\mathcal{A}}dx_2dx_1 - \int_{-1}^{1}\int_{10}^{11} \sqrt{\mathcal{A}}dx_2dx_1 \geqslant \big(c + o(1)\big)\|\hat{u}\|_{\mathcal{H}}, \tag{132}$$

where $c > 0$ is a numerical constant. Moreover, by Lemma 5.1 (rough decay estimate) we know that

$$\|\hat{u}\|_{\mathcal{H}} \geqslant e^{\delta\tau}. \tag{133}$$

Hence, there is some $(a, b) \in (-1, 1) \times (0, 1)$ (depending on τ) such that

$$\sqrt{\mathcal{A}(a, b, \tau)} - \sqrt{\mathcal{A}(a, b + 10, \tau)} \geqslant ce^{\delta\tau}. \tag{134}$$

Thus, the concavity of $\sqrt{\mathcal{A}}$ yields the desired result, as this implies

$$\mathcal{A}(a, b + Ce^{-\delta\tau}, \tau) = 0, \tag{135}$$

for some constant $C < \infty$. This proves the corollary. □

6. Bubble-sheet analysis in the unstable mode

In this section, we consider ancient noncollapsed flows with a bubble sheet tangent flow at $-\infty$, whose unstable mode is dominant. We will show that each such flow has some nonvanishing expansion parameters associated to it. This is the content of the fine bubble-sheet theorem (Theorem 6.7) and the nonvanishing expansion theorem (Theorem 6.9).

More precisely, throughout this section we assume the tilted rescaled flow $\tilde{M}^{X_0}_\tau$ around some point X_0, has a dominant unstable mode, i.e.

$$U_0 + U_- \leqslant C\rho^{-1} U_+. \tag{136}$$

6.1. Analysis of the untilted flow.

When dealing with the unstable mode case, it is convenient to work with the renormalized flow $\bar{M}^{X_0}_\tau$ itself, and not with its tilted version $\tilde{M}^{X_0}_\tau$. Our first task is to show that when dealing with the *untilted* renormalized flow around *any* point X, the unstable mode is still the dominant one.

Given any point X, for $\tau \ll 0$ (depending only on $Z(X)$) we can write $\bar{M}^X_\tau$ as a graph of a function $\bar{u}(\cdot,\tau)$ over $\Gamma \cap B_{2\rho(\tau)}(0)$, where $\rho = \rho^X(\cdot,\tau)$, which satisfies (47) and (49). We will work with the truncated function

$$\check{u} = \bar{u}\chi(r/\rho(\tau)). \tag{137}$$

Proposition 6.1 (evolution of truncated graph function)**.** *There exists some universal constant $C < \infty$ such that*

$$\|(\partial_\tau - \mathcal{L})\check{u}\|_{\mathcal{H}} \leqslant C\rho^{-1}\|\check{u}\|_{\mathcal{H}} \tag{138}$$

for $\tau \ll 0$.

Proof. This follows from repeating the argument in Lemma 4.5 and Proposition 4.6 with $A(\tau) = 0$. □

Now, the functions

$$\begin{aligned} \check{U}_+(\tau) &:= \|P_+\check{u}(\cdot,\tau)\|^2_{\mathcal{H}}, \\ \check{U}_0(\tau) &:= \|P_0\check{u}(\cdot,\tau)\|^2_{\mathcal{H}}, \\ \check{U}_-(\tau) &:= \|P_-\check{u}(\cdot,\tau)\|^2_{\mathcal{H}}, \end{aligned} \tag{139}$$

satisfy the evolution inequalities (99). Applying the Merle-Zaag ODE-Lemma (Lemma B.1), we infer that there are universal constants $C, R < \infty$ (where we can assume that $R > 10^3$) such that for every X either

$$\check{U}_+ + \check{U}_- = o(\check{U}_0), \tag{140}$$

or

$$\check{U}_0 + \check{U}_- \leqslant C\rho^{-1}\check{U}_+ \tag{141}$$

whenever $\rho \geqslant R$.

Proposition 6.2 (dominant mode)**.** *The unstable mode is dominant for the untilted flow. More precisely, the inequality* (141) *holds for every X.*

Proof. Let us first show that the statement hold for $X = X_0$. By assumption, the tilted flow with center X_0 has dominant unstable mode, i.e.

$$U_0 + U_- \leqslant C\rho U_+. \tag{142}$$

Together with the evolution inequalities (99) this also implies

$$\frac{d}{d\tau}U_+(\tau) \geqslant (1 - C\rho^{-1})U_+. \tag{143}$$

Note that (143) implies

$$\frac{d}{d\tau}e^{-\frac{9}{10}\tau}U_+(\tau) \geqslant 0, \tag{144}$$

for sufficiently large $-\tau$. Hence,

$$U = (1 + o(1))U_+ \leqslant Ce^{\frac{9}{10}\tau}. \tag{145}$$

Therefore, Lemma 4.7 yields that

$$|S^{X_0}(\tau) - I| \leqslant Ce^{\frac{9}{20}\tau}. \tag{146}$$

Combining (145) and (146) we infer that the untilted flow satisfies

$$\check{U} \leqslant Ce^{\frac{9}{20}\tau}. \tag{147}$$

On the other hand, if we had $\check{U}_+ + \check{U}_- = o(\check{U}_0)$, then arguing as in Lemma 5.1 we would see that $\check{U} \geqslant e^{\delta\tau}$ for every $\delta > 0$ and $-\tau$ sufficiently large, which is inconsistent with (147). Thus, for $X = X_0$, we indeed get

$$\check{U}_0 + \check{U}_- \leqslant C\rho^{-1}\check{U}_+. \tag{148}$$

Finally, any neck centered at a general point X merges with the neck centered at X_0 as $\tau \to -\infty$. Thus, (141) holds for every X. □

Recapping, for any center X we therefore have

$$\check{U}_0 + \check{U}_- \leqslant C\rho^{-1}\check{U}_+, \tag{149}$$

and, thanks to the evolution inequalities (99), also

$$\frac{d}{d\tau}\check{U}_+ \geqslant (1 - C\rho^{-1})\,\check{U}_+, \tag{150}$$

whenever $\rho \geqslant R$, where $R \geqslant 10^3$ and C are some universal constants. Increasing R further, we can also assume that $R \geqslant 10C$.

6.2. **Graphical radius.** The goal of this section is to prove Proposition 6.4, which gives a lower bound for the optimal graphical radius.

We now some $\varepsilon < 1/R$ in the definition of the bubble-sheet scale (Definition 2.6). In what follows, $C < \infty$ and $\mathcal{T} > -\infty$ will denote constants that are allowed to change from line to line, but depend *only* on an upper bound of $Z(X)$. In particular, our initial choice of graphical radius satisfies $\rho^X(\tau) \geqslant R$ for $\tau \in (-\infty, \mathcal{T}]$ and (149) and (150) hold for all $\tau \in (-\infty, \mathcal{T}]$.

Lemma 6.3 (decay estimate)**.** *There exists some constant $C < \infty$, depending only on an upper bound for $Z(X)$, such that*

$$\|\check{u}(\cdot, \tau)\|_{\mathcal{H}} \leqslant Ce^{\frac{9}{20}\tau} \tag{151}$$

and

$$\|\check{u}(\cdot, \tau)\|_{C^{10}(\{r \leqslant 100\})} \leqslant Ce^{\frac{9}{20}\tau} \tag{152}$$

hold for $\tau \leqslant \mathcal{T}$.

Proof. Since $\bar{M}^X_{\mathcal{T}}$ is an ε-graph over $\Gamma \cap B_{2\rho^X}(0)$, we get $\|\check{u}(\cdot, \mathcal{T})\|_{\mathcal{H}} \leqslant 1$. Note that (150) implies $\frac{d}{d\tau}(e^{-\frac{9}{10}\tau}\check{U}_+) \geqslant 0$ for $\tau \leqslant \mathcal{T}$. Hence,

$$e^{-\frac{9}{10}\tau}\check{U}_+ \leqslant e^{-\frac{9}{10}\mathcal{T}}\|\check{u}(\cdot, \mathcal{T})\|^2_{\mathcal{H}} \leqslant C. \tag{153}$$

This gives the first inequality (151). The second inequality (152) follows from (151) by parabolic estimates (see Theorem A.1 for details). □

Proposition 6.4 (improved graphical radius)**.** *There exists some $\mathcal{T} > -\infty$, depending only on an upper bound for $Z(X)$, such that*

$$\bar{\rho}(\tau) = e^{-\frac{1}{9}\tau} \tag{154}$$

is a graphical radius function satisfying (47) *and* (49) *for $\tau \leqslant \mathcal{T}$.*

Proof. We recall from [ADS19, Thm. 8.2] that the profile function u_a of the ADS-barriers satisfies

$$\sqrt{2}\left(1 - \frac{x^2}{a^2}\right) \leqslant u_a(x) \leqslant \sqrt{2}\left(1 - \frac{x^2 - 10}{1000a^2}\right) \tag{155}$$

for every $x \in [2, a]$, provided that a is large enough.

The upper bound for u_a and Proposition 6.3 (decay estimate) imply that $\{r = 10\} \cap \Gamma_a$ is enclosed by $\bar{M}^X_\tau$ for $a^2 \leqslant \frac{1}{C}e^{-\frac{9}{20}\tau}$. Since Γ_a is enclosed by $\bar{M}^X_\tau$ for $\tau \ll 0$, the maximum principle thus guarantees that

$$\bar{M}^X_\tau \text{ encloses } \{r \geqslant 10\} \cap \Gamma_a \text{ for } a^2 \leqslant Ce^{-\frac{9}{20}\tau}. \tag{156}$$

Hence, by the estimate (152), the convexity of M^X_τ, and the lower bound for u_a, we get that M^X_τ is graphical over the cylinder up to a radius $\frac{1}{C}e^{-\frac{9}{40}\tau}$, with C^0-norm less than $Ce^{-\frac{9}{40}\tau}$. Moreover, using quantitative differentiation (see [CHN13]), as in the proof of [CHH18, Proposition 4.1], we see that all such points with radius less than $\frac{1}{C}e^{-\frac{9}{40}\tau}$, have regularity scale bounded uniformly from above and below. Interpolating this and the C^0-estimate, we see that up to a radius of $2e^{-\frac{1}{9}\tau}$, the hypersurface M^X_τ is a C^1-graph over the cylinder with norm less than $\varepsilon_0 e^{-\frac{9}{80}\tau}$. Hence, using standard parabolic Schauder estimates for the renormalized flow around any point within that radius $e^{-\frac{1}{9}\tau}$ of the point X yields the desired result. □

From now on, we work with $\rho = \bar{\rho}$ from Proposition 6.4 (improved graphical radius), and in particular define $\check{u}$, $\check{U}_0$, $\check{U}_\pm, \ldots$ with respect to this improved graphical radius. Note that the unstable mode is still dominant.

Corollary 6.5 (sharp decay estimate)**.** *There exist constants $C < \infty$ and $\mathcal{T} > -\infty$, depending only an upper bound for $Z(X)$, such that*

$$\|\check{u}\|_{\mathcal{H}} \leqslant Ce^{\tau/2}, \tag{157}$$

and

$$\|\check{u}(\cdot,\tau)\|_{C^{10}(\{r\leqslant 100\})} \leqslant Ce^{\tau/2}. \tag{158}$$

holds for $\tau \leqslant \mathcal{T}$.

Proof. Combining the evolution inequality (150), Lemma 6.3 (decay estimate) and Proposition 6.4 (improved graphical radius) yields

$$\frac{d}{d\tau}\left(e^{-\tau}\check{U}_+\right) \geqslant -Ce^{-\tau+\frac{1}{9}\tau+\frac{9}{10}\tau} = -Ce^{\frac{1}{90}\tau} \tag{159}$$

for all $\tau \leqslant \mathcal{T}$. Thus, $\check{U}_+ \leqslant Ce^\tau$. Since the unstable mode is dominant, this proves that $\|\check{u}\|_{\mathcal{H}} \leqslant Ce^{\frac{\tau}{2}}$.
Moreover, Proposition 6.4 (improved graphical radius), inequality (149) and Proposition 6.1 (evolution of truncated graph function) then give

$$\check{U}_0 + \check{U}_- + \|(\partial_\tau - \mathcal{L})\check{u}\|^2_{\mathcal{H}} \leqslant Ce^{\frac{10}{9}\tau}. \tag{160}$$

Using this, (158) follows from parabolic estimates (see Theorem A.1 for details). This proves the corollary. □

6.3. **The fine bubble-sheet theorem.** The goal of this section is to prove Theorem 6.7. Recalling the basis of $\mathcal{H}_+$ from (93) we can write

$$P_+\check{u}^X = a^X_0(\tau) + a^X_1(\tau)x_1 + a^X_2(\tau)x_2 + a^X_3(\tau)\cos\theta + a^X_4(\tau)\sin\theta, \tag{161}$$

where the superscript is to remind us that all these quantities can (a priori) depend on the center point X.

Proposition 6.6 (estimate for coefficients)**.** *The coefficients defined in* (161) *satisfy the estimates*

$$|a^X_0(\tau)| \leqslant Ce^{\frac{11}{18}\tau}, \tag{162}$$

and

$$\sum_{i=1}^{4}\left|e^{-\frac{\tau}{2}}a_i^X(\tau)-\bar{a}_i^X\right|\leqslant Ce^{\frac{1}{9}\tau}, \tag{163}$$

for $\tau\leqslant\mathcal{T}$, where $\bar{a}_i^X$ are numbers that might depend on X.

Proof. Letting $\check{E}:=(\partial_\tau-\mathcal{L})\check{u}$, and using $\mathcal{L}\,1=1$, we compute

$$\frac{d}{d\tau}a_0^X(\tau)=(\tfrac{e}{2\pi})^{\frac14}\int(\mathcal{L}\check{u}+\check{E})\tfrac{1}{4\pi}e^{-\frac{|x|^2}{4}}=a_0^X(\tau)+(\tfrac{e}{2\pi})^{\frac14}\int\tfrac{\check{E}}{4\pi}e^{-\frac{|x|^2}{4}}. \tag{164}$$

Hence, using Lemma 6.1, Proposition 6.4 and Corollary 6.5, we obtain

$$\left|\frac{d}{d\tau}\left(e^{-\tau}a_0^X(\tau)\right)\right|\leqslant Ce^{-\tau}\|\check{E}\|_{\mathcal{H}}\leqslant Ce^{-\tau+\frac{\tau}{2}+\frac{\tau}{9}}\leqslant Ce^{-\frac{7}{18}\tau}. \tag{165}$$

Integrating this from τ to $\mathcal{T}$ implies (162).
In a similar manner, using $\mathcal{L}x_i=\frac12 x_i$ for $i=1,\dots,4$ we get

$$\left|\frac{d}{d\tau}\left(e^{-\frac{\tau}{2}}a_i^X(\tau)\right)\right|\leqslant Ce^{-\frac{\tau}{2}}\|\check{E}\|_{\mathcal{H}}\leqslant Ce^{\frac{\tau}{9}}, \tag{166}$$

so integrating from $-\infty$ to τ yields (163) with

$$\bar{a}_i^X=\lim_{\tau\to-\infty}e^{-\tau/2}a_i^X(\tau). \tag{167}$$

This proves the proposition. ◻

Theorem 6.7 (The fine bubble-sheet theorem)**.** *Let $\{M_t\}$ be an ancient noncollapsed flow in $\mathbb{R}^4$, with a bubble sheet tangent flow at $-\infty$, whose unstable mode is dominant. Then there exists some constants $a_1,\cdots,a_4$, independent of the center point $X=(x_0^1,x_0^2,x_0^3,x_0^4,t_0)$, such that*

$$\bar{a}_1^X=a_1,\qquad \bar{a}_2^X=a_2,\qquad \bar{a}_3^X=a_3-x_0^3,\qquad \bar{a}_4^X=a_4-x_0^4. \tag{168}$$

Moreover, for every center point X the truncated graph function $\check{u}^X(\cdot,\tau)$ of the renormalized flow $\bar{M}_\tau^X$ satisfies the estimates

$$\left\|\check{u}^X-e^{\frac{\tau}{2}}\left(a_1x_1+a_2x_2+\bar{a}_3^X\cos\theta+\bar{a}_4^X\sin\theta\right)\right\|_{\mathcal{H}}\leqslant Ce^{\frac59\tau}, \tag{169}$$

and

$$\left\|\check{u}^X-e^{\frac{\tau}{2}}\left(a_1x_1+a_2x_2+\bar{a}_3^X\cos\theta+\bar{a}_4^X\sin\theta\right)\right\|_{L^\infty(\{r\leqslant100\})}\leqslant Ce^{\frac{19}{36}\tau}. \tag{170}$$

for $\tau\leqslant\mathcal{T}$, where $C<\infty$ and $\mathcal{T}>-\infty$ only depend on an upper bound on the bubble sheet scale $Z(X)$.

Proof. Consider the difference

$$D^X:=\check{u}^X-e^{\tau/2}\left(\bar{a}_1^Xx_1+\bar{a}_2^Xx_2+\bar{a}_3^X\cos\theta+\bar{a}_4^X\sin\theta\right). \tag{171}$$

Using Proposition 6.6 (estimate for coefficients) we see that

$$|D^X|\leqslant|\check{u}^X-P_+\check{u}^X|+C(|x|+1)e^{\frac{11}{18}\tau}. \tag{172}$$

Since by (160) we have $\check{U}_0 + \check{U}_- \leqslant Ce^{\frac{10}{9}\tau}$, it follows that

$$\|D^X\|_{\mathcal{H}} \leqslant Ce^{\frac{5}{9}\tau}, \tag{173}$$

which proves (169) modulo the claim about the coefficients.

Next, we observe that

$$\|D^X\|_{L^2(\{r\leqslant 100\})} \leqslant C\|D^X\|_{\mathcal{H}} \leqslant Ce^{\frac{5}{9}\tau}, \tag{174}$$

and, using Corollary 6.5 (sharp decay estimate), that

$$\|\nabla^3 D^X\|_{L^2(\{r\leqslant 100\})} \leqslant C\|\hat{u}^X\|_{C^3(\{r\leqslant 100\})} + Ce^{\frac{\tau}{2}} \leqslant Ce^{\frac{\tau}{2}}. \tag{175}$$

Applying Agmon's inequality this yields

$$\|D^X\|_{L^\infty(\{r\leqslant 100\})} \leqslant C\|D^X\|^{\frac{1}{2}}_{L^2(\{r\leqslant 100\})}\|D^X\|^{\frac{1}{2}}_{H^3(\{r\leqslant 100\})} \leqslant Ce^{\frac{19}{36}\tau}. \tag{176}$$

Finally, let us show that the parameters $a_1, \cdots, a_4$, defined via (168), are independent of X.

First, let us show that they are independent of time translation. Denote by $\check{u}^{X'}$ the function obtained by considering the renormalized mean curvature flow with center $X' = (x_0^1, x_0^2, x_0^3, x_0^4, 0)$. A direct calculation shows that for $\theta \in (0, 2\pi]$, $x_1^2 + x_2^2 \leqslant 100$ we have

$$\begin{aligned}\check{u}^X\left(\frac{x_1}{\sqrt{1+t_0e^\tau}}, \frac{x_2}{\sqrt{1+t_0e^\tau}}, \theta, \tau - \log(1+t_0e^\tau)\right)&\\ = \frac{1}{\sqrt{1+t_0e^\tau}}(\sqrt{2} + \check{u}^{X'}(x_1, x_2, \theta, \tau)) - \sqrt{2}.&\end{aligned} \tag{177}$$

This implies

$$\|\check{u}^X - \check{u}^{X'}\|_{L^\infty(\{r\leqslant 100\})} = o(e^{\tau/2}), \tag{178}$$

and thus together with (176) yields $\bar{a}_i^X = \bar{a}_i^{X'}$ for every $i = 1, \dots 4$.

Now, comparing the renormalized flows with center $X' = (x_0^1, x_0^2, x_0^3, x_0^4, 0)$ and center $X'' = ((0,0,0,0),0)$, we need to relate both the parameters of the functions $\check{u}^{X'}$ and $\check{u}^{X''}$ which describe the same point in the original flow, and the distance of such a point from the respective axii. This leads to

$$\begin{aligned}\sqrt{2} + \check{u}^{X'}(x_1 - x_0^1e^{\tau/2}, x_2e - x_0^2e^{\tau/2}, \theta + O(e^{\tau/2}), \tau)&\\ = \mathrm{dist}\left(\left(\sqrt{2} + \check{u}^{X''}(x_1, x_2, \theta, \tau)\right)(\cos\theta, \sin\theta), e^{\tau/2}\left(x_0^3, x_0^4\right)\right).&\end{aligned} \tag{179}$$

By Taylor expansion and Corollary 6.5 we have

$$\begin{aligned}\mathrm{dist}\left(\left(\sqrt{2} + \check{u}^{X''}(x_1, x_2, \theta, \tau)\right)(\cos\theta, \sin\theta), e^{\tau/2}\left(x_0^3, x_0^4\right)\right)&\\ = \sqrt{2} + \check{u}^{X''}(x_1, x_2, \theta, \tau) - x_0^3\cos\theta e^{\tau/2} - x_0^4\sin\theta e^{\tau/2} + o(e^{\tau/2}).&\end{aligned} \tag{180}$$

Together with (176), the above formulas imply that

$$\bar{a}_1^{X'} = \bar{a}_1^{X''}, \quad \bar{a}_2^{X'} = \bar{a}_2^{X''}, \quad \bar{a}_3^{X'} = \bar{a}_3^{X''} - x_0^3, \quad \bar{a}_4^{X'} = \bar{a}_4^{X''} - x_0^4. \tag{181}$$

This finishes the proof of the theorem. □

6.4. **The nonvanishing expansion theorem.** Our next goal it to show that a_1 and a_2 can not simultaneously vanish.

We decompose $\mathcal{H}_+$ into $\mathcal{H}_{1/2} = \mathrm{span}\{x_1, x_2, \cos\theta, \sin\theta\}$ and $\mathcal{H}_1 = \mathrm{span}\{1\}$. Also, we define $P_{1/2}$ and P_1 as the projections to $\mathcal{H}_{1/2}$ and $\mathcal{H}_1$, respectively. In addition, we denote $\check{U}_{1/2} = \|P_{1/2}\check{u}\|^2_{\mathcal{H}}$ and $\check{U}_1 = \|P_1\check{u}\|^2_{\mathcal{H}}$.

Lemma 6.8 (decay if coefficients vanished)**.** *If the center X was such that $\bar{a}_1^X = \ldots = \bar{a}_4^X = 0$, then there would be a constant $K_0 > 0$ such that*

$$\check{U}_1 = K_0 e^{2\tau}(1 + O(e^{\frac{1}{9}\tau})), \tag{182}$$

and moreover we would have

$$\check{U}_{1/2} + \check{U}_0 + \check{U}_- \leqslant Ce^{\frac{1}{9}\tau}\check{U}_1. \tag{183}$$

Proof. Assuming $\bar{a}_1^X = \ldots = \bar{a}_4^X = 0$, Proposition 6.6 implies

$$a_0^X(\tau)^2 + \ldots + a_4^X(\tau)^2 \leqslant Ce^{\frac{11}{18}\tau}, \tag{184}$$

hence

$$\check{U}_1 + \check{U}_{1/2} \leqslant e^{\frac{11}{9}\tau}. \tag{185}$$

Moreover, since the unstable mode is dominant, we have

$$\check{U}_0 + \check{U}_- \leqslant Ce^{\frac{1}{9}\tau}(\check{U}_1 + \check{U}_{1/2}). \tag{186}$$

Now, using Proposition 6.1 we get the evolution inequalities

$$\left|\tfrac{d}{d\tau}\check{U}_{1/2} - \check{U}_{1/2}\right| \leqslant Ce^{\frac{1}{9}\tau}(\check{U}_{1/2} + \check{U}_1), \tag{187}$$

$$\left|\tfrac{d}{d\tau}\check{U}_1 - 2\check{U}_1\right| \leqslant Ce^{\frac{1}{9}\tau}(\check{U}_{1/2} + \check{U}_1). \tag{188}$$

Applying the Merle-Zaag ODE-lemma (Lemma B.1) with $U_0 = e^{-\tau}\check{U}_{1/2}$, $U_- = 0$, and $U_+ = e^{-\tau}\check{U}_1$, we get either $\check{U}_1 = o(\check{U}_{1/2})$ or $\check{U}_{1/2} \leqslant Ce^{\frac{1}{9}\tau}\check{U}_1$. In the former case, arguing as in Lemma 5.1 we could infer that $e^{-\tau}\hat{U}_{1/2} \geqslant e^{\frac{1}{9}\tau}$ for every $-\tau$ sufficiently large, contradicting (185). Hence,

$$\check{U}_{1/2} \leqslant Ce^{\frac{1}{9}\tau}\check{U}_1. \tag{189}$$

Together with (186) this proves the estimate (183).
Moreover, using (189), our differential inequality takes the form

$$\left|\tfrac{d}{d\tau}\check{U}_1 - 2\check{U}_1\right| \leqslant Ce^{\frac{1}{9}\tau}\check{U}_1, \tag{190}$$

which can be rewritten as

$$\left|\tfrac{d}{d\tau}(\log(e^{-2\tau}\check{U}_1))\right| \leqslant Ce^{\frac{1}{9}\tau}. \tag{191}$$

Integrating this from $-\infty$ to τ gives that there exists some constant $\tilde{K}_0$ such

$$|\log(e^{-2\tau}U_1) - \tilde{K}_0| \leqslant Ce^{\frac{1}{9}\tau} \tag{192}$$

for all $\tau \leqslant \mathcal{T}$. Exponentiating both sides and using the approximation $e^x \approx 1 + x$ for small $x \in \mathbb{R}$ give (182). In particular, $K_0 = e^{\tilde{K}_0} > 0$. □

Theorem 6.9 (the nonvanishing expansion theorem)**.** *The coefficients from the fine-bubble sheet theorem satisfy* $|a_1| + |a_2| > 0$*.*

Proof. Suppose towards a contradiction that $a_1 = a_2 = 0$. Then, we can choose a point X such that $\bar{a}_1^X = \ldots = \bar{a}_4^X = 0$. By Lemma 6.8 and parabolic estimates (see Theorem A.1) we get

$$\|u(\cdot,\tau)\|_{C^4(\{r\leqslant 100\})} \leqslant Ce^{\tau}, \tag{193}$$

and

$$x_3^2 + x_4^2 = 2(1 + Ke^{\tau}) + o(e^{\tau}) \tag{194}$$

on $\{r \leqslant 100\}$, where $K = (2e/\pi)^{1/4}K_0$. Hence, the rescaled flow with the center $X' = X + (0, K)$ satisfies

$$x_3^2 + x_4^2 = o(e^{\tau}), \tag{195}$$

uniformly on $\{r \leqslant 100\}$, see also [ADS19, Lemma 5.11] for a more detailed explanation. Since we re-centered by shifting only in time direction, the new point X' still satisfies $\bar{a}_1^{X'} = \ldots = \bar{a}_4^{X'} = 0$, so Lemma 6.8 gives some $K_0' > 0$ such that

$$e^{-2\tau}\check{U}_1 = K_0' + O(e^{\frac{1}{9}\tau}). \tag{196}$$

Since $K_0' \neq 0$, this contradicts (195). This proves the theorem. □

7. Conclusion in the unstable mode case

The goal of this section is to prove the following theorem.

Theorem 7.1 (unstable mode)**.** *The only noncompact ancient noncollapsed flow in* $\mathbb{R}^4$*, with bubble-sheet tangent flow at* $-\infty$*, whose unstable mode is dominant, is* $\mathbb{R} \times 2d$*-bowl.*

Proof. By the reduction from Section 2 it is enough to prove that if the unstable mode is dominant, then its blowdown contains a line.

So let $M_t = \partial K_t$ be a noncompact ancient noncollapsed flow in $\mathbb{R}^4$, with bubble-sheet tangent flow at $-\infty$, whose unstable mode is dominant, and suppose towards a contradiction that its blowdown $\check{K}$ does not contain a line. Then the flow is strictly convex, and $\check{K}$ is a halfline or a wedge of angle less than π in $\mathbb{R}^2 \times \{0\}$. Choosing suitable coordinates we can assume that $\check{K}$ is symmetric across the x_1-axis, and is contained in the half space $\{x_1 \geqslant 0\}$. By translating, we may also assume that $0 \in M_0$ is the point in M_0 with smallest x_1-value. This implies that for every $h > 0$ there exist a unique point $x_h^\pm \in M_0 \cap \{x_1 = h\}$ at which x_2 is maximized/minimized.

By the fine bubble-sheet theorem (Theorem 6.7) and the non-vanishing expansion theorem (Theorem 6.9) there exists expansion parameters a_1, a_2 associated to our flow such that $|a_1| + |a_2| > 0$.

Claim 7.2 (bubble-sheet scale)**.** There exists some constant $C < \infty$ such that

$$\sup_h Z(x_h^\pm) \leqslant C. \tag{197}$$

Proof of the claim. We will argue similarly as in the proofs of [CHH18, Proposition 5.8] and [CHHW19, Proposition 6.2].
Suppose towards a contradiction that $Z(x_{h_i}^\pm) \to \infty$ for some sequence $\{h_i\}$ with $\lim_{i\to\infty} h_i = \infty$. Let $\mathcal{M}^i$ be the sequence of flows obtained by shifting $x_{h_i}^\pm$ to the origin, and parabolically rescaling by $Z(x_{h_i}^+)^{-1}$. By [HK17a, Thm. 1.14] we can pass to a subsequential limit $\mathcal{M}^\infty$, which is an ancient noncollapsed flow that is weakly convex and smooth until it becomes extinct. Note also that $\mathcal{M}^\infty$ has bubble-sheet tangent flow at $-\infty$.

We next observe that, $\mathcal{M}^\infty$ cannot be a round shrinking $\mathbb{R}^2 \times S^1$. Indeed, if such a cylinder became extinct at time 0 that would contradict the definition of the bubble-sheet scale, and if it became extinct at some later time that would contradict the fact that $M_0^\infty \cap (\mathbb{R}^2 \times \{0\})$ is a strict subset of $\mathbb{R}^2 \times \{0\}$ by construction.
Thus, by Theorem 4.8 (Merle-Zaag alternative) for the flow $\mathcal{M}^\infty$ either the neutral mode is dominant or the unstable mode is dominant. If the neutral mode is dominant, then for large i, this contradicts the fact that $\mathcal{M}^i$ has dominant unstable mode. If the unstable mode is dominant, then by the fine-bubble sheet theorem (Theorem 6.7) and the non-vanishing expansion theorem (Theorem 6.9) the limit $\mathcal{M}^\infty$ has some expension parameters a_1^∞, a_2^∞ that do not vanish simultaneously. However, this contradicts the fact that the expansion parameters of $\mathcal{M}^i$ are obtained from the expansion parameters (a_1, a_2) of $\mathcal{M}$ by scaling by $Z(x_{h_i}^+)^{-1} \to 0$. This concludes the proof of the claim. □

Continuing the proof of the theorem, let $h_i \to \infty$ and consider the sequence $\mathcal{M}^i := \mathcal{M} - (x_{h_i}^+, 0)$, which is obtained by translating in space time without rescaling. Taking a subsequential limit, Claim 7.2 implies that this limit $\mathcal{M}^\infty$ is an ancient noncollapsed flow with a bubble-sheet tangent at $-\infty$. Moreover, arguing as in the proof of Claim 7.2 we see that $\mathcal{M}^\infty$ has a dominant unstable mode, with the same expansion parameters a_1, a_2 as $\mathcal{M}$.

On the other hand, by the choice of $x_{h_i}^+$, we have

$$\frac{x_{h_i}^+}{\|x_{h_i}^+\|} \to w^+ \in \partial\check{K}, \tag{198}$$

with $\langle w^+, e_2\rangle \geqslant 0$. Thus, $\mathcal{M}^\infty$ splits off a line in the direction w^+. Therefore, by [BC19] the limit $\mathcal{M}^\infty$ is $\mathbb{R}$ times a 2-dimensional bowl, where the $\mathbb{R}$-factor is in the direction w^+, and where the translation direction v^+ is the orthogonal complement of w^+ in $\mathbb{R}^2 \times \{0\}$ with

$$\langle v^+, e_2 \rangle < 0. \tag{199}$$

As the expansion parameters of $\mathcal{M}^\infty$ are also a_1, a_2 we see by observation (or by the fine-neck theorem from [CHH18]) that

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \gamma v^+ \tag{200}$$

for some $\gamma > 0$. Combining this with (199), we get that $a_2 < 0$.

Arguing similarly using $x^-_{h_i}$ gives that $a_2 > 0$; a contradiction. This concludes the proof of the theorem. □

Appendix A. Local L^∞-estimate

In this appendix, we consider the renormalized mean curvature flow given (in some ball) as a graph over the cylinder $\Gamma = \mathbb{R}^{n-d} \times S^d(\sqrt{2d})$, namely our variables are $(y, \sqrt{2d}\,\omega) \in \Gamma$ where $y \in \mathbb{R}^{n-d}$ and $\omega \in S^d$. Let

$$\mathcal{L}u = \Delta_\Gamma u - \tfrac{1}{2} x^{\mathrm{tau}} \cdot \nabla_\Gamma u + u = \rho^{-1}\mathrm{div}(\rho \nabla u) + \tfrac{1}{2d}\Delta_{S^d} u + u, \tag{201}$$

where $\rho(y)$ is the Gaussian density given by

$$\rho(y) = (4\pi)^{-\frac{d+1}{2}} e^{-\frac{d}{2}} e^{-\frac{|y|^2}{4}}. \tag{202}$$

Given $R \gg 1$, we let $\Gamma_R = \{(y, \omega) \in \Gamma : |y| \leqslant R\}$ and $Q(R) = \Gamma_R \times [-R^2, 0]$. We consider smooth solutions $u : Q(2R) \to \mathbb{R}$ to the equation

$$u_\tau = \mathcal{L}u + E. \tag{203}$$

Theorem A.1 (c.f. [Lie96, Theorem 6.17])**.** *Suppose that for some constants $C_0, k < \infty$, the error E satisfies*

$$|E| \leqslant C_0(|u| + |\nabla u|) + k. \tag{204}$$

Then,

$$\sup_{Q(R)} |u| \leqslant C\left[k + \left(\int_{Q(2R)} u^2 \, d\mathrm{vol}_\Gamma d\tau\right)^{\frac{1}{2}}\right], \tag{205}$$

where $C = C(C_0, R, n) < \infty$.

Proof. Instead of u, we consider the function $\bar{u} = \rho u$ which solves

$$\bar{u}_\tau = \mathrm{div}\big(\rho \nabla_\Gamma(\rho^{-1}\bar{u})\big) + \rho E. \tag{206}$$

Since this equation satisfies the conditions of [Lie96, Theorem 6.17], given $q > 2$ we can obtain the following inequality as the proof in Lieberman[1]

$$\int_\Omega \bar{u}^{q-2}|\nabla \bar{u}|^2 v^{\alpha q-n-2}\xi^2 d\mathrm{vol}_\Gamma d\tau \leqslant \frac{Cq^2}{R^2}\int_\Omega \bar{u}^q v^{\alpha q-n-2} d\mathrm{vol}_\Gamma d\tau, \tag{207}$$

where $\Omega = Q(2R) \cap \{\bar{u} \geqslant kR\}$, $\xi = (1 - |x|^2/4R^2)_+(1 + t/4R^2)_+$ is a cut-off function, $v = (1 - kR/\bar{u})_+ \in [0,1)$, and $\alpha = (n+2)/2$. Thus, $h = \bar{u}v^\alpha$ satisfies

$$\int_\Omega |\nabla h|^2 \leqslant Cq^4 R^{-2}\int_\Omega h^2 v^{-2} \tag{208}$$

Hence, the Sobolev type inequality from Lemma A.2 below yields

$$\begin{aligned} Cq^4\int_\Omega h^2 v^{-2} &\geqslant \left(\int_\Omega |\nabla h|^2 + \int_\Omega |h|^2\right) + \int_\Omega |h|^2 \\ &\geqslant \frac{1}{C}\left(\int_\Omega |\nabla h|^2 + |h|^2\right)^{\frac{n}{n+2}}\left(\int_\Omega h^2\right)^{\frac{2}{n+2}} \geqslant \frac{1}{C}\left(\int_\Omega h^{\frac{2(n+2)}{n}}\right)^{\frac{2}{n+2}} \end{aligned} \tag{209}$$

for some constant $C = C(n,R) < \infty$. Thus, setting $\kappa = (n+2)/n$, $w = \bar{u}v^\alpha$, and $d\mu = R\xi(1 - kR/\bar{u})_+^{-n-2} d\mathrm{vol}_\Gamma d\tau$ gives

$$\left(\int_\Omega w^{\kappa q} d\mu\right)^{1/\kappa q} \leqslant C^{1/q} q^{4/q}\left(\int_\Omega w^q d\mu\right)^{1/q}. \tag{210}$$

Hence, iterating this process with $q = 2\kappa^j$ for $j \in \mathbb{N}$ yields the result. □

The following lemma has been used in lieu of [Lie96, Theorem 6.9]:

Lemma A.2. *Suppose that $u \in C^\infty(Q(R))$ is a non-negative function satisfying $u = 0$ on $\partial\Gamma_R \times [-R^2, 0]$. Then, there exists some constant $C = C(n,R) < \infty$ such that*

$$\int_{Q(R)} u^{\frac{2(n+2)}{n}} \leqslant C\left(\int_{Q(R)} u^2\right)^{\frac{2}{n}}\left(\int_{Q(R)} |\nabla u|^2 + u^2\right). \tag{211}$$

Proof. The Hölder inequality yields

$$\int_Q u^{\frac{2(n+2)}{n}} \leqslant \left(\int_Q u^2\right)^{\frac{1}{n}}\left(\int_Q u^{\frac{2(n+1)}{n-1}}\right)^{\frac{n-1}{n}}. \tag{212}$$

Applying the Michael-Simon inequality this implies

$$\int_Q u^{\frac{2(n+2)}{n}} \leqslant C\left(\int_Q u^2\right)^{\frac{1}{n}}\left(\int_Q u^{\frac{n+2}{n}}|Du| + u^{\frac{2(n+1)}{n}}\right)^{\frac{n-1}{n}} \tag{213}$$

Using again Hölder's inequality we obtain the desired result. □

[1] The book states a stronger inequality, which is wrong, but easily correctable. Here, we provide the necessary modification of the argument for convenience of the reader.

Appendix B. Merle-Zaag ODE-lemma

We recall the following variant of the Merle-Zaag ODE-lemma [MZ98, Lemma A.1], which has been proved in [CM19, Lemma B.1]:

Lemma B.1. *U_0, U_+, $U_- : (-\infty, 0] \to \mathbb{R}$ are absolutely continuous non-negative functions satisfying $U_0 + U_+ + U_- > 0$ and*

$$\liminf_{s\to-\infty} U_-(s) = 0. \tag{214}$$

Suppose that there exist some constant $c_0 > 0$ and positive increasing function $\sigma : (-\infty, 0] \to \mathbb{R}$ such that $\lim_{s\to-\infty} \sigma(s) = 0$ and the following hold

$$|U_0'| \leqslant \sigma(U_0 + U_- + U_+), \tag{215}$$

$$U_-' \leqslant -c_0 U_- + \sigma(U_0 + U_+), \tag{216}$$

$$U_+' \geqslant c_0 U_+ - \sigma(U_0 + U_-). \tag{217}$$

Then, there exists $c = c(c_0) > 0$ and $\delta = \delta(c_0) > 0$ such that if $\sigma(s_0) < \delta$ then

$$U_- \leqslant c\sigma(U_0 + U_+) \text{ on } (-\infty, s_0], \tag{218}$$

and either

$$U_+ \leqslant o(U_0), \tag{219}$$

or

$$U_0 \leqslant c\sigma U_+ \tag{220}$$

for all $s \leqslant s_0$.

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