HITTING ESTIMATES ON EINSTEIN MANIFOLDS AND APPLICATIONS

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Abstract. We generalize the Benjamini-Pemantle-Peres estimate relating hitting probability and Martin capacity to the setting of manifolds with nonnegative Ricci curvature. As applications we obtain: (1) a sharp estimate for the probability that Brownian motion comes close to the high curvature part of a Ricci-flat manifold, (2) a proof of an unpublished theorem of Naber that every noncollapsed limit of Ricci-flat manifolds is a weak solution of the Einstein equations, (3) an effective intersection estimate for two independent Brownian motions on manifolds with nonnegative Ricci curvature and positive asymptotic volume ratio.

1. Introduction

By a classical theorem of Kakutani [Kak44], Brownian motion in Euclidean space of dimension $n \geq 3$ hits a compact set with positive probability if and only if the set has positive Newtonian capacity. More recently, Benjamini-Pemantle-Peres [BPP95] discovered a sharp two-sided estimate for the hitting probability. One of their key insights was that in order to get the right dependence on the initial point x_0, one should work with the capacity associated to the Martin kernel

$$K(x,y) = \frac{|x-x_0|^{n-2}}{|x-y|^{n-2}}\,. \tag{1.1}$$

Our first main result (see Theorem 1.1 below) generalizes the Benjamini-Pemantle-Peres estimate to the setting of Riemannian manifolds (M^n, g) with nonnegative Ricci-curvature, i.e.

$$\mathrm{Ric} \geq 0. \tag{1.2}$$

Let $p_t(x,y)$ be the heat kernel on M. By Cheeger-Yau [CY81] it satisfies the lower bound

$$p_t(x,y) \geq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}}\,, \tag{1.3}$$

and by Li-Yau [LY86] for each $\gamma > 4$ there is a $C_\gamma = C_\gamma(n) < \infty$ such that

$$p_t(x,y) \leq \frac{C_\gamma \omega_n}{(4\pi)^{n/2}\mathrm{Vol}(B(x,\sqrt{t}))} e^{-\frac{d(x,y)^2}{\gamma t}}\,. \tag{1.4}$$

Let X_t be Brownian motion on M starting at $x_0 \in M$ (for background see e.g. [Hsu02]). In terms of the heat kernel, X_t is uniquely characterized by the formula

$$\mathbb{P}_{x_0}[X_{t_1} \in U_1, \ldots, X_{t_k} \in U_k] = \int_{U_1 \times \ldots \times U_k} p_{t_1}(x_0, y_1) \cdots p_{t_k - t_{k-1}}(y_{k-1}, y_k) dV(y_1) \ldots dV(y_k). \tag{1.5}$$

For any Borel set $A \subseteq M$ the Martin capacity is defined by

$$\mathrm{Cap}_K(A) := \left[\inf_{\nu(A)=1} \iint_{A \times A} K(x,y)\, d\nu(x) d\nu(y)\right]^{-1}, \tag{1.6}$$

where K denotes the Martin kernel

$$K(x,y)=\begin{cases}\left(\dfrac{d(x_0,y)}{d(x,y)}\right)^{n-2} & \text{if } x\neq y\\ \infty & \text{if } x=y\end{cases}\,. \tag{1.7}$$

Using these notions, we can now state our hitting estimate:

Theorem 1.1. *Let (M^n,g) be a complete Riemannian manifold of dimension $n\geq 3$, with nonnegative Ricci-curvature. Then for all compact sets $A\subseteq M^n$ and all $T\leq\infty$ we have the hitting estimate*

$$\frac{1}{2\Lambda}\mathrm{Cap}_K(A)\leq \mathbb{P}_{x_0}[X_t\in A \textit{ for some } 0<t<T]\leq \Lambda\mathrm{Cap}_K(A), \tag{1.8}$$

where

$$\Lambda=\frac{\omega_n}{v}\left(\frac{\gamma}{4}\right)^{\frac{n}{2}-1}C_\gamma\frac{\int_0^\infty \xi^{\frac{n}{2}-2}e^{-\xi}d\xi}{\int_{\frac{\mathrm{diam}(A\cup\{x_0\})^2}{4T}}^\infty \xi^{\frac{n}{2}-2}e^{-\xi}d\xi}, \tag{1.9}$$

with

$$v=\inf_{y\in A\cup\{x_0\}}\frac{\mathrm{Vol}(B(y,\sqrt{2T}))}{(2T)^{n/2}}. \tag{1.10}$$

Remark 1.2.

(i) If $M=\mathbb{R}^n$ is flat Euclidean space, then we can choose $\gamma=4$, $C_\gamma=1, v=\omega_n$ and $T=\infty$, and hence recover the sharp estimate

$$\frac{1}{2}\mathrm{Cap}_K(A)\leq \mathbb{P}_{x_0}[B_t\in A \text{ for some } 0<t<\infty]\leq \mathrm{Cap}_K(A)$$

from Benjamini-Pemantle-Peres [BPP95].

(ii) In practice, one usually chooses $T\gg \mathrm{diam}(A\cup\{x_0\})^2$ so that $\Lambda\approx\frac{\omega_n}{v}\left(\frac{\gamma}{4}\right)^{\frac{n}{2}-1}C_\gamma$. In particular, if (M,g) has nonvanishing asymptotic volume ratio one can simply choose $T=\infty$.

(iii) Assuming $\mathrm{Vol}(B(x_0,r_0))\geq v_0r_0^n$, by Bishop-Gromov volume comparison we of course have

$$v\geq\left(\frac{r_0}{\max\{\sqrt{2T},r_0+\mathrm{diam}(A\cup\{x_0\})\}}\right)^n v_0\,.$$

One of the main features of Theorem 1.1, in contrast to prior hitting estimates on manifolds in the literature (see e.g. [CF86, GSC02]), is that our estimate is universal in the sense that it only depends on a lower bound for the noncollapsing constant. Let us now discuss three applications.

Our first application concerns the question: How likely is it that Brownian motion comes close to the almost singular part of a Ricci-flat manifold? To discuss this, recall that for $x\in M$ the regularity scale is defined by

$$\mathrm{reg}(x):=\sup\big\{r\leq 1\,:\,\sup_{B(x,r)}|\mathrm{Rm}|\leq r^{-2}\big\}. \tag{1.11}$$

We consider the ε-singular set

$$S_\varepsilon:=\{x\in M\,:\,\mathrm{reg}(x)\leq\varepsilon\}, \tag{1.12}$$

and its ε-tubular neighborhood

$$N_\varepsilon(S_\varepsilon) = \{x \in M \,:\, d(x, S_\varepsilon) \leq \varepsilon\}. \tag{1.13}$$

Similarly, we consider the Wiener sausage

$$N_\varepsilon(X[0,T]) = \{x \in M \,:\, d(x, X_t) \leq \varepsilon \text{ for some } 0 \leq t \leq T\}\,. \tag{1.14}$$

One can think of the Wiener sausage as the path swept out by a Brownian particle of size ε. The following theorem estimates the probability that this particle hits the almost singular part $N_\varepsilon(S_\varepsilon)$:

Theorem 1.3. *Let (M^n, g) be a complete Riemannian Ricci-flat manifold of dimension $n \geq 3$. Then for all $0 < \varepsilon \leq 1$, $1 \leq T < \infty$ and all $x_0 \in M$ with $d(x_0, S_\varepsilon) \geq 3\varepsilon$ we have the estimate*

$$\mathbb{P}_{x_0}\left[N_\varepsilon(X[0,T]) \cap N_\varepsilon(S_\varepsilon) \cap B(x_0, \sqrt{T}) \neq \emptyset\right] \leq \frac{C\varepsilon^2}{d(x_0, S_\varepsilon)^2}, \tag{1.15}$$

where $C = C(n, v) < \infty$ only depends on the dimension, and a lower bound v for $\frac{\mathrm{Vol}(B(x_0,\sqrt{T}))}{T^{n/2}}$.

The estimate in Theorem 1.3 is sharp as illustrated by the following proposition:

Proposition 1.4. *If $(M, g) = (T^*S^2 \times \mathbb{R}^{n-4}, g_\varepsilon + \delta)$, where g_ε is the Eguchi-Hanson metric with central two-sphere of size ε, then for all $x_0 \in M$ with $3\varepsilon \leq d(x_0, S_\varepsilon) \leq \sqrt{T}/3$ we have the lower bound*

$$\mathbb{P}_{x_0}\left[X[0,T] \cap S_\varepsilon \cap B(x_0, \sqrt{T}) \neq \emptyset\right] \geq \frac{c\varepsilon^2}{d(x_0, S_\varepsilon)^2}. \tag{1.16}$$

Our second application concerns weak solutions of the Einstein equations. To discuss this, recall that Naber [Nab13] discovered a characterization of Ricci-flat metrics via a sharp gradient estimate on path space. Specifically, he proved that a Riemannian manifold (M^n, g) solves the Einstein equations

$$\mathrm{Ric} = 0 \tag{1.17}$$

if and only if the gradient estimate

$$|\nabla_x \mathbb{E}_x[F]| \leq \mathbb{E}_x[|\nabla^{||} F|] \tag{1.18}$$

holds for all test functions $F : PM \to \mathbb{R}$ on path space and almost every $x \in M$. Here, $\mathbb{E}_x$ denotes the expectation with respect to the Wiener measure of Brownian motion starting at $x \in M$, and $\nabla^{||} F(X) \in T_xM$ denotes the parallel gradient, which for cylinder functions $F(X) = f(X_{t_1}, \ldots, X_{t_k})$ is defined by

$$\nabla^{||} F(X) = \sum_{i=1}^{k} P_{t_i}(X) \nabla^{(i)} f(X_{t_1}, \ldots, X_{t_k})\,, \tag{1.19}$$

where $P_t(X) : T_{X_t}M \to T_xM$ denotes stochastic parallel transport. Motivated by this, Naber then introduced a notion of weak solutions of the Einstein equations for metric measure spaces. To describe this, let us first impose the convention that all metric measure spaces (M, d, m) are assumed to be locally compact, complete, length spaces such that m is a locally finite, σ-finite

Borel measure with $\mathrm{spt}(m) = M$. Now, recall that on any metric measure space (M, d, m) one has the Cheeger-energy

$$E[u] = \tfrac{1}{2}\int_M |\nabla u|_*^2\, dm, \tag{1.20}$$

where $|\nabla u|_*$ denotes the minimal relaxed gradient [AGS14]. If E satisfies the parallelogram identity, then M is called Riemannian. Any Riemannian metric measure space has a linear heat flow and can be equipped with a Wiener measure of Brownian motion characterized by (1.5), see [Fuk80].

Definition 1.5 (c.f. Naber [Nab13])**.** A Riemannian metric measure space (M, d, m) is called a weak solution of the Einstein equations if the gradient estimate

$$|\nabla_x \mathbb{E}_x[F]| \leq \mathbb{E}_x[|\nabla^{\|} F|] \tag{1.21}$$

holds for all test functions $F : PM \to \mathbb{R}$ on path space and m-almost every $x \in M$, where ∇_x is the local Lipschitz slope and the parallel gradient is the one constructed in [Nab13, Section 14].

Using Theorem 1.3 we give a proof of the following unpublished result of Naber:[1]

Theorem 1.6 (Naber)**.** *Every noncollapsed limit of Ricci-flat manifolds is a weak solution of the Einstein equations.*

For comparison, recall that by the theory of Lott-Villani [LV09] and Sturm [Stu06] lower bounds on Ricci-curvature are preserved under weak limits. While lower curvature bounds survive even in the collapsing case, for upper curvature bounds the noncollapsing condition is essential.

Our final application concerns the question: How likely is it that two independent Brownian motions come close to each other? For Brownian motion in Euclidean space of dimension $n > 4$ by work of Aizenman [Aiz85] and Pemantle-Peres-Shapiro [PPS96] the probability that two independent Brownian particles of size ε starting apart hit each other is proportional to ε^{n-4}. The following theorem generalizes this to manifolds with nonnegative Ricci curvature with nonvanishing asymptotic volume ratio:

Theorem 1.7. *For any $n > 4$ and $v > 0$ there exists a constant $C = C(n, v) < \infty$ with the following significance. For any complete Riemannian manifold (M, g) of dimension $n > 4$ with $\mathrm{Ric} \geq 0$ and $\lim_{r\to\infty} \frac{\mathrm{Vol}(B(x,r))}{r^n} \geq v$, for any two points $x_0, \tilde{x}_0 \in M$ with $d(x_0, \tilde{x}_0) > 5\varepsilon$ we have*

$$C^{-1}\frac{\varepsilon^{n-4}}{d(x_0,\tilde{x}_0)^{n-4}} \leq \mathbb{P}_{x_0,\tilde{x}_0}[N_\varepsilon(X[0,\infty)) \cap N_\varepsilon(\tilde{X}[0,\infty)) \neq \emptyset] \leq C\frac{\varepsilon^{n-4}}{d(x_0,\tilde{x}_0)^{n-4}}. \tag{1.22}$$

Again the estimate is universal in the sense that it only depends on a lower bound for the asymptotic volume ratio.

This article is organized as follows: In Section 2, we prove the hitting estimate. To do so, we generalize the approach by Benjamini-Pemantle-Peres [BPP95] to the manifold setting, using in particular the heat kernel estimates from Cheeger-Yau and Li-Yau. The upper bound is derived via an entrance time decomposition, while the lower bound relies on a second moment estimate. In Section 3, we prove the theorems from our three applications. To prove Theorem 1.3, we have

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[1] RH thanks Aaron Naber for pointing out in 2014 that this follows from the solution of the codimension 4 conjecture by Cheeger-Naber [CN15], which in particular implies that the singular set has zero capacity.

to estimate the Martin capacity of the almost singular set $N_\varepsilon(S_\varepsilon)$. A key ingredient for this is the work of Jiang-Naber [JN16]. To prove Theorem 1.6, we show that almost every path of Brownian motion starting at a regular point stays in the regular part, and then conclude as in [Nab13]. To prove Theorem 1.7, we have to estimate the expected Martin capacity of the Wiener sausage. We do this via suitable dyadic decompositions and repeated applications of Theorem 1.1.

Acknowledgements. The second author has been supported by an NSERC Discovery Grant and a Sloan Research Fellowship.

2. Proof of the hitting estimate

Proof of Theorem 1.1. To prove the upper bound, we consider an entrance time decomposition. To this end, let

$$\tau := \inf\{t > 0 \,:\, X_t \in A\} \in [0,\infty] \tag{2.1}$$

be the first hitting time of A. Note that $\tau \wedge T$ is a stopping time. Let μ be the distribution of $X_{\tau\wedge T}$, i.e.

$$\mu(A') = \mathbb{P}_{x_0}[X_{\tau\wedge T} \in A']. \tag{2.2}$$

Observe that

$$\mathbb{P}_{x_0}[X_t \in A \text{ for some } 0 < t < T] = \mu(A). \tag{2.3}$$

Now, for any Borel set $A' \subseteq A$ we consider the expected occupancy time

$$\mathbb{E}_{x_0}\left[\int_0^{2T} 1_{\{X_t\in A'\}} dt\right] = \int_0^{2T}\int_{A'} \rho_t(x_0,y)\, dV(y)\, dt\,. \tag{2.4}$$

By the strong Markov property, $X'_t = X_{\tau\wedge T+t}$ is a Brownian motion starting from $X_{\tau\wedge T}$. Using this and the lower heat kernel bound (1.3) from Cheeger-Yau we can estimate

$$\begin{aligned}
\mathbb{E}_{x_0}\left[\int_0^{2T} 1_{\{X_t\in A'\}} dt\right] &\geq \mathbb{E}_{x_0}\left[\int_{\tau\wedge T}^{2T} 1_{X_t\in A'} dt\right]\\
&= \int_M \mathbb{E}_x\left[\int_0^{2T-(\tau\wedge T)} 1_{X'_t\in A'} dt\right] d\mu(x)\\
&= \int_M \int_{A'} \int_0^{(2T-\tau)\vee T} \rho_t(x,y)\, dt\, dV(y)\, d\mu(x)\\
&\geq \int_A \int_{A'} \int_0^{T} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}}\, dt\, dV(y)\, d\mu(x).
\end{aligned} \tag{2.5}$$

On the other hand, by the upper heat kernel bound (1.4) from Li-Yau we can estimate

$$\mathbb{E}_{x_0}\left[\int_0^{2T} 1_{\{X_t\in A'\}} dt\right] \leq \int_{A'}\int_0^{2T} \frac{\omega_n}{v}\frac{C_\gamma}{(4\pi t)^{n/2}} e^{-\frac{d(x_0,y)^2}{\gamma t}}\, dt\, dV(y)\,. \tag{2.6}$$

Since A' was arbitrary, combining the above inequalities we infer that

$$\int_A \int_0^T \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(x,y)^2}{4t}}\, dt\, d\mu(x) \leq \int_0^{2T} \frac{\omega_n}{v} \frac{C_\gamma}{(4\pi t)^{n/2}} e^{-\frac{d(x_0,y)^2}{\gamma t}}\, dt \tag{2.7}$$

for dV-almost every $y \in A$. Via substitution, we can rewrite this in the form

$$\int_A \int_{\frac{d(x,y)^2}{4T}}^\infty \frac{\xi^{\frac{n}{2}-2}}{d(x,y)^{n-2}} e^{-\xi}\, d\xi\, d\mu(x) \leq \frac{\omega_n}{v}\left(\frac{\gamma}{4}\right)^{\frac{n}{2}-1} C_\gamma \int_{\frac{d(x,y)^2}{2\gamma T}}^\infty \frac{\xi^{\frac{n}{2}-2}}{d(x_0,y)^{n-2}} e^{-\xi}\, d\xi\,. \tag{2.8}$$

Recalling the definition of the Martin kernel (1.7), this shows that

$$\int_A K(x,y)\, d\mu(x) \leq \frac{\omega_n}{v}\left(\frac{\gamma}{4}\right)^{\frac{n}{2}-1} C_\gamma \left[\frac{\int_0^\infty \xi^{\frac{n}{2}-2} e^{-\xi} d\xi}{\int_{\frac{\mathrm{diam}(A)^2}{4T}}^\infty \xi^{\frac{n}{2}-2} e^{-\xi} d\xi}\right]. \tag{2.9}$$

At first, this only follows for dV-almost every $y \in A$, but considering the regularized kernel $K_\varepsilon(x,y) = \left(\frac{d(x_0,y)}{d(x,y)+\varepsilon}\right)^{n-2}$ and sending $\varepsilon \to 0$ we see that (2.9) in fact holds for every $y \in A$. Finally, integrating (2.9) with respect to $d\mu(y)$ we conclude that

$$\int_{A\times A} K(x,y)\, d\mu(x)\, d\mu(y) \leq \Lambda \mu(A). \tag{2.10}$$

Recalling (2.3), this proves the upper bound

$$\mathbb{P}_{x_0}[X_t \in A \text{ for some } 0 < t < T] \leq \Lambda \mathrm{Cap}_K(A). \tag{2.11}$$

To prove the lower bound, we use a second moment estimate. By general properties of Cap_K, we can assume without loss of generality that $x_0 \notin A$.[2] Given any probability measure ν on A, let us consider the random variable

$$Z_\varepsilon := \int_A \frac{\int_0^T 1_{\{X_t \in B(y,\varepsilon)\}} dt}{\mathbb{E}_{x_0}[\int_0^T 1_{\{X_t \in B(y,\varepsilon)\}} dt]}\, d\nu(y). \tag{2.12}$$

By construction, we have $\mathbb{E}_{x_0}[Z_\varepsilon] = 1$. The random variable Z_ε measures a weighted occupancy time of the ε-tubular neighborhood of $N_\varepsilon(A)$, in particular

$$\begin{aligned} \mathbb{P}_{x_0}[X_t \in N_\varepsilon(A) \text{ for some } 0 < t < T] &\geq \mathbb{P}_{x_0}[Z_\varepsilon > 0] \\ &\geq \frac{1}{\mathbb{E}_{x_0}[Z_\varepsilon^2]}, \end{aligned} \tag{2.13}$$

where we used the Cauchy-Schwarz inequality in the last step. With the goal of deriving an upper bound for $\mathbb{E}_{x_0}[Z_\varepsilon^2]$, let us define the functions

$$h_\varepsilon(x,y) := \mathbb{E}_x\left[\int_0^T 1_{\{X_t \in B(y,\varepsilon)\}}\, dt\right], \tag{2.14}$$

and

$$h_\varepsilon^*(x,y) := \sup_{z \in B(x,\varepsilon)} h_\varepsilon(z,y). \tag{2.15}$$

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[2] Otherwise, consider $A \setminus B(0,1/i)$ and use that $\lim_{i\to\infty} \mathrm{Cap}_K(A \setminus B(x_0,1/i)) = \mathrm{Cap}_K(A)$.

Using the strong Markov property we can now estimate

$$\begin{aligned}\mathbb{E}_{x_0}[Z_\varepsilon^2] &= 2\mathbb{E}_{x_0}\left[\int_A\int_A\int_0^T\int_s^T \frac{1_{\{X_t\in B(y,\varepsilon)\}\cap\{X_s\in B(x,\varepsilon)\}}}{h_\varepsilon(x_0,y)h_\varepsilon(x_0,x)}\,dt\,ds\right]d\nu(x)\,d\nu(y)\\ &\le 2\mathbb{E}_{x_0}\left[\int_A\int_A\int_0^T 1_{\{X_s\in B(x,\varepsilon)\}}\frac{h_\varepsilon^*(x,y)}{h_\varepsilon(x_0,y)h_\varepsilon(x_0,x)}\right]ds\,d\nu(x)\,d\nu(y)\\ &= 2\int_A\int_A \frac{h_\varepsilon^*(x,y)}{h_\varepsilon(x_0,y)}\,d\nu(x)\,d\nu(y).\end{aligned}\tag{2.16}$$

To proceed, we need the following claim.

Claim 2.1. *We have*

$$\lim_{\varepsilon\searrow 0}\int_A\int_A\frac{h_\varepsilon^*(x,y)}{h_\varepsilon(x_0,y)}\,d\nu(x)\,d\nu(y)=\int_A\int_A\frac{\int_0^T\rho_t(x,y)\,dt}{\int_0^T\rho_t(x_0,y)\,dt}\,d\nu(x)\,d\nu(y).\tag{2.17}$$

Proof of Claim 2.1. First observe that

$$\lim_{\varepsilon\searrow 0}\frac{h_\varepsilon(x_0,y)}{\mathrm{Vol}(B(y,\varepsilon))}=\int_0^T\rho_t(x_0,y)\,dt \quad\text{and}\quad \lim_{\varepsilon\searrow 0}\frac{h_\varepsilon^*(x,y)}{\mathrm{Vol}(B(y,\varepsilon))}=\int_0^T\rho_t(x,y)\,dt.$$

Hence, we have the pointwise convergence

$$\lim_{\varepsilon\searrow 0}\frac{h_\varepsilon^*(x,y)}{h_\varepsilon(x_0,y)}=\frac{\int_0^T\rho_t(x,y)\,dt}{\int_0^T\rho_t(x_0,y)\,dt}.\tag{2.18}$$

To take the limit under the integral it suffices to find a constant $C<\infty$ such that

$$\frac{h_\varepsilon^*(x,y)}{h_\varepsilon(x_0,y)}\le C\frac{\int_0^T\rho_t(x,y)\,dt}{\int_0^T\rho_t(x_0,y)\,dt}\tag{2.19}$$

for $\varepsilon\le\frac{1}{2}d(x_0,A)\wedge T^{1/2}$. To this end, first observe that using the heat kernel estimates (1.3) and (1.4) from Cheeger-Yau and Li-Yau, similarly as in (2.8), we obtain

$$C^{-1}d(x,y)^{2-n}\le\int_0^T\rho_t(x,y)\,dt\le Cd(x,y)^{2-n}\tag{2.20}$$

for all $x,y\in N_\varepsilon(A)\cup\{x_0\}$, where C depends on n, v, T and $\mathrm{diam}(A\cup\{x_0\})$. Therefore, the ratio

$$\frac{h_\varepsilon(x_0,y)}{\mathrm{Vol}(B(y,\varepsilon))}=\frac{1}{\mathrm{Vol}(B(y,\varepsilon))}\int_{B(y,\varepsilon)}\int_0^T\rho_t(x_0,z)\,dt\,dV(z)\tag{2.21}$$

is uniformly bounded from above and below by positive constants. Thus, it suffices to show

$$\frac{h_\varepsilon^*(x,y)}{\mathrm{Vol}(B(y,\varepsilon))}\le C\int_0^T\rho_t(x,y)\,dt.\tag{2.22}$$

This clearly holds if $d(x,y)\geq 3\varepsilon$. On the other hand, if $d(x,y)<3\varepsilon$, then using the coarea formula and Bishop-Gromov volume comparison, for z with $d(x,z)<\varepsilon$ we compute

$$\begin{aligned}\int_{B(y,\varepsilon)}\int_0^T \rho_t(z,\zeta)\,dt\,dV(\zeta) &\leq \int_{B(z,5\varepsilon)}\int_0^T \rho_t(z,\zeta)\,dt\,dV(\zeta)\\ &\leq C\int_0^{5\varepsilon}\int_{\partial B(z,r)} d(z,\zeta)^{2-n}d\mathcal{H}^{n-1}(\zeta)\,dr\\ &\leq C\mathrm{Vol}(B(y,\varepsilon))\int_0^T \rho_t(x,y)\,dt.\end{aligned}\tag{2.23}$$

This finishes the proof of the claim. □

Continuing the proof of the theorem, by estimate (2.16) and Claim 2.1 we have

$$\limsup_{\varepsilon\searrow 0}\mathbb{E}_{x_0}[Z_\varepsilon^2]\leq 2\int_A\int_A \frac{\int_0^T\rho_t(x,y)\,dt}{\int_0^T\rho_t(x_0,y)\,dt}\,d\nu(x)\,d\nu(y).\tag{2.24}$$

Since

$$\frac{\int_0^T\rho_t(x,y)\,dt}{\int_0^T\rho_t(x_0,y)\,dt}\leq\frac{\omega_n}{v}\left(\frac{\gamma}{4}\right)^{\frac{n}{2}-1}C_\gamma\left[\frac{\int_0^\infty \xi^{\frac{n}{2}-2}e^{-\xi}\,d\xi}{\int_{\frac{\sup_{y\in A}d(x_0,y)^2}{4T}}^\infty \xi^{\frac{n}{2}-2}e^{-\xi}\,d\xi}\right]K(x,y),\tag{2.25}$$

we infer that

$$\limsup_{\varepsilon\searrow 0}\mathbb{E}_{x_0}[Z_\varepsilon^2]\leq 2\Lambda\int_A\int_A K(x,y)\,d\nu(x)\,d\nu(y).\tag{2.26}$$

Together with (2.13) and the continuity of Brownian paths this shows that

$$\mathbb{P}_{x_0}[X_t\in A\text{ for some }0<t<T]\geq\frac{1}{2\Lambda}\left[\int_A\int_A K(x,y)\,d\nu(x)\,d\nu(y)\right]^{-1}.\tag{2.27}$$

Since the probability measure ν was arbitrary, this proves the lower bound

$$\mathbb{P}_{x_0}[X_t\in A\text{ for some }0<t<T]\geq\frac{1}{2\Lambda}\mathrm{Cap}_K(A),\tag{2.28}$$

which concludes the proof of the theorem. □

3. APPLICATIONS

3.1. **Probability to come close to almost singular part.**

In this section, we prove Theorem 1.3. Recall that for any metric space (X,d), the α-dimensional Hausdorff measure is defined by

$$\mathcal{H}^\alpha(A)=\lim_{\varepsilon\searrow 0}\mathcal{H}^\alpha_\varepsilon(A),\tag{3.1}$$

where

$$\mathcal{H}^\alpha_\varepsilon(A):=\inf\left\{\sum_i \mathrm{diam}(A_i)^\alpha\,:\,A\subseteq\bigcup_i A_i\text{ and }\mathrm{diam}(A_i)\leq\varepsilon\right\}.\tag{3.2}$$

We will need the following lemma relating capacity and Hausdorff measure.

Lemma 3.1. *Let (X,d) be a metric space and $A \subseteq X$ be a Borel set. Then*

$$\mathcal{H}^{\alpha}_{\varepsilon}(A) \geq \frac{\mu(A)^2}{\iint_{\{d(x,y)\leq\varepsilon\}} d(x,y)^{-\alpha} d\mu(x) d\mu(y)} \tag{3.3}$$

for any $\alpha > 0$ and any nonzero finite (Borel) measure μ on A.

Proof of Lemma 3.1. If A_i is a countable disjoint covering of A whose diameters are bounded by ε, then

$$\begin{aligned} \mu(A)^2 &\leq \left[\sum_i \mathrm{diam}(A_i)^{\alpha/2} \mathrm{diam}(A_i)^{-\alpha/2} \mu(A_i)\right]^2 \\ &\leq \sum_i \mathrm{diam}(A_i)^{\alpha} \sum_i \mathrm{diam}(A_i)^{-\alpha} \mu(A_i)^2 \\ &\leq \sum_i \mathrm{diam}(A_i)^{\alpha} \sum_i \int_{A_i}\int_{A_i} d(x,y)^{-\alpha} d\mu(x) d\mu(y) \\ &\leq \sum_i \mathrm{diam}(A_i)^{\alpha} \iint_{\{d(x,y)\leq\varepsilon\}} d(x,y)^{-\alpha} d\mu(x) d\mu(y). \end{aligned} \tag{3.4}$$

This proves the lemma. ☐

We can now prove the main result of this section:

Proof of Theorem 1.3. First observe that

$$N_{\varepsilon}(X[0,T]) \cap N_{\varepsilon}(S_{\varepsilon}) \neq \emptyset \quad \Rightarrow \quad X_t \in N_{2\varepsilon}(S_{\varepsilon}) \text{ for some } 0 < t < T. \tag{3.5}$$

Hence, using Theorem 1.1 we can estimate

$$\mathbb{P}_{x_0}[N_{\varepsilon}(X[0,T]) \cap N_{\varepsilon}(S_{\varepsilon}) \cap B(x_0,\sqrt{T}) \neq \emptyset] \leq \Lambda \mathrm{Cap}_K(N_{2\varepsilon}(S_{\varepsilon}) \cap B(x_0,\sqrt{T})). \tag{3.6}$$

To proceed, let us write Cap_N for the capacity associated to the Newtonian kernel

$$N(x,y) = \frac{1}{d(x,y)^{n-2}}\,. \tag{3.7}$$

In general, the Martin capacity and the Newtonian capacity can be compared via

$$\frac{\mathrm{Cap}_N(A)}{\sup_{x\in A} d(x_0,x)^{n-2}} \leq \mathrm{Cap}_K(A) \leq \frac{\mathrm{Cap}_N(A)}{\inf_{x\in A} d(x_0,x)^{n-2}} \tag{3.8}$$

To use this efficiently, let us chop up our set into dyadic annuli. Namely, set $r_i := 2^i d(x_0, N_{2\varepsilon}(S_{\varepsilon}))$, let k be the smallest integer such that $r_k \geq \sqrt{T}$, and estimate

$$\begin{aligned} \mathrm{Cap}_K(N_{2\varepsilon}(S_{\varepsilon}) \cap B(x_0,\sqrt{T})) &\leq \sum_{i=1}^{k} \mathrm{Cap}_K(N_{2\varepsilon}(S_{\varepsilon}) \cap (B(x_0,r_i) \setminus B(x_0,r_{i-1}))) \\ &\leq \sum_{i=1}^{k} \frac{1}{r_{i-1}^{n-2}} \mathrm{Cap}_N(N_{2\varepsilon}(S_{\varepsilon}) \cap B(x_0,r_i)). \end{aligned} \tag{3.9}$$

Now by the deep work of Jiang-Naber [JN16, Theorem 1.8], we have

$$\mathrm{Vol}(N_{2\varepsilon}(S_{\varepsilon}) \cap B(x_0,r_i)) \leq C\varepsilon^4 r_i^{n-4}, \tag{3.10}$$

where C only depends on the dimension n, and a lower bound v for $\mathrm{Vol}(B(x_0,\sqrt{T}))/T^{n/2}$. Hence, we can find $J_i \leq C\varepsilon^{4-n}r_i^{n-4}$ points $x_{i,j}\in N_{2\varepsilon}(S_\varepsilon)\cap B(x_0,r_i)$, such that

$$N_{2\varepsilon}(S_\varepsilon)\cap B(x_0,r_i)\subseteq \bigcup_{j=1}^{J_i} B(x_{i,j},\varepsilon)\,. \tag{3.11}$$

This yields

$$\mathrm{Cap}_N(N_{2\varepsilon}(S_\varepsilon)\cap B(x_0,r_i)) \leq J_i \max_j \mathrm{Cap}_N(B(x_{i,j},\varepsilon)) \leq C\varepsilon^2 r_i^{n-4}, \tag{3.12}$$

where we also used that by Lemma 3.1 the Newtonian capacity of ε-balls can be estimated by

$$\mathrm{Cap}_N(B(x,\varepsilon)) \leq \mathcal{H}^{n-2}_\varepsilon(B(x,\varepsilon)) \leq C\varepsilon^{n-2}. \tag{3.13}$$

Putting things together we conclude that

$$\mathbb{P}_{x_0}[N_\varepsilon(X[0,T])\cap N_\varepsilon(S_\varepsilon)\cap B(x_0,\sqrt{T})\neq\emptyset] \leq C\sum_{i=1}^{k}\frac{1}{r_{i-1}^{n-2}}\varepsilon^2 r_i^{n-4} \leq C\frac{\varepsilon^2}{r_0^2}. \tag{3.14}$$

This proves the theorem. □

Finally, let us show that the estimate is sharp by considering the example of the Eguchi-Hanson metric [EH79].

Proof of Proposition 1.4. Let $r_0 := d(x_0,S_\varepsilon)$. In view of the lower bound from Theorem 1.1 it is enough to show that

$$\mathrm{Cap}_K(S_\varepsilon\cap B(x_0,3r_0)) \geq c\varepsilon^2 r_0^{-2}. \tag{3.15}$$

To begin with, by (3.8) we have

$$\mathrm{Cap}_K(S_\varepsilon\cap B(x_0,3r_0)) \geq (3r_0)^{2-n}\mathrm{Cap}_N(S_\varepsilon\cap B(x_0,3r_0)). \tag{3.16}$$

Next, observe that $S_\varepsilon\cap B(x_0,3r_0)$ contains a set of the form

$$Z := B^4(p,\varepsilon)\times B^{n-4}(q,2r_0) \tag{3.17}$$

where p is any point on the central two-sphere of the Eguchi-Hanson manifold, and $q\in\mathbb{R}^{n-4}$. Letting ν be the uniform probability measure on Z, an elementary computation shows that

$$\int_Z\int_Z \frac{d\nu(x)\,d\nu(y)}{d(x,y)^{n-2}} = \frac{1}{\mathrm{Vol}(Z)^2}\int_Z\int_Z\frac{dV(x)\,dV(y)}{d(x,y)^{n-2}} \leq \frac{C\varepsilon^2}{\mathrm{Vol}(Z)} \leq \frac{C}{\varepsilon^2 r_0^{n-4}}. \tag{3.18}$$

Hence,

$$\mathrm{Cap}_N(Z) \geq C^{-1}\varepsilon^2 r_0^{n-4}. \tag{3.19}$$

Putting things together, this proves the proposition. □

3.2. **Weak solutions of the Einstein equations.** In this section we prove that every noncollapsed limit of Ricci-flat manifolds is a weak solution of the Einstein equations.

Proof of Theorem 1.6. Let (M, d, m) be a noncollapsed limit of Ricci-flat manifolds. The key is to establish the following claim.

Claim 3.1. *For every regular point $x_0 \in M$ almost every path of Brownian motion starting at x_0 stays in the regular part of M.*

Proof of the claim. Let $S \subseteq M$ be the singular set. Since a countable union of null events is a null event it is enough to show that for every $0 < R < \infty$ and every $T < \infty$ we have

$$\mathbb{P}_{x_0}[X_t \in S \cap B(x_0, R) \text{ for some } 0 < t < T] = 0. \tag{3.20}$$

Let $S_\varepsilon \subseteq M$ be the quantitative singular set. For any positive integer $N < \infty$ consider the partition $t_k = kT/N$, where $k = 1, \ldots, N$. By Lemma 3.2 (see below) we can estimate

$$\begin{aligned}\mathbb{P}_{x_0}&[X_t \in S_\varepsilon \cap B(x_0, R) \text{ for some } 0 < t < T] \\ &\leq \mathbb{P}_{x_0}[X_{t_k} \in N_\varepsilon(S_\varepsilon) \cap B(x_0, R) \text{ for some } 1 \leq k \leq N] + Ne^{-\frac{\varepsilon^2 N}{100}}.\end{aligned} \tag{3.21}$$

By assumption we have the pointed measured Gromov-Hausdorff convergence $(M_i, g_i, dV_i) \to (M, d, m)$. Using the local regularity theorem [And90] and the convergence of the heat kernel [Din02] we can estimate

$$\begin{aligned}\mathbb{P}_{x_0}&[X_{t_k} \in N_\varepsilon(S_\varepsilon) \cap B(x_0, R) \text{ for some } 1 \leq k \leq N] \\ &\leq \limsup_{i\to\infty} \mathbb{P}_{x_0^i}[B^i_{t_k} \in N_{2\varepsilon}(S^i_{2\varepsilon}) \cap B(x_0^i, R+1) \text{ for some } 1 \leq k \leq N].\end{aligned} \tag{3.22}$$

By Theorem 1.3 this is bounded by $C\varepsilon^2$ for ε small enough. Hence,

$$\mathbb{P}_{x_0}[X_t \in S_\varepsilon \cap B(x_0, R) \text{ for some } 0 < t < T] \leq C\varepsilon^2 + Ne^{-\frac{\varepsilon^2 N}{100}}. \tag{3.23}$$

Sending $N \to \infty$ this yields

$$\mathbb{P}_{x_0}[X_t \in S_\varepsilon \cap B(x_0, R) \text{ for some } 0 < t < T] \leq C\varepsilon^2. \tag{3.24}$$

Sending $\varepsilon \to 0$ we conclude that

$$\mathbb{P}_{x_0}[X_t \in S \cap B(x_0, R) \text{ for some } 0 < t < T] = 0\,. \tag{3.25}$$

This proves the the claim. □

Having established Claim 3.1, the computation from [Nab13, Section 6.2] (or alternatively the simplified computation from [HN18]) now goes through and shows that

$$|\nabla_x \mathbb{E}_x[F]| \leq \mathbb{E}_x[|\nabla^{||} F|]\,. \tag{3.26}$$

This proves the theorem. □

In the above proof we used the following lemma:

Lemma 3.2. *For any geodesic ball $B(x_0, r)$ properly contained in a (possibly incomplete) manifold with nonnegative Ricci curvature, denoting by τ the first exit time, for every $\delta \leq \frac{1}{2n} r^2$ we have*

$$\mathbb{P}_{x_0}[\tau \leq \delta] \leq e^{-\frac{r^2}{100\delta}}. \tag{3.27}$$

Proof. Write $r(x) := d(x, x_0)$ and consider the radial process $r_t := r(X_t)$. By a classical result of Kendall [Ken87] this satisfies the evolution equation

$$dr_t = d\beta_t + \tfrac{1}{2}\Delta r(X_t)\, dt - dL_t, \tag{3.28}$$

where β_t is a one-dimensional Euclidean Brownian motion and L_t is a nondecreasing process (which increases only on the cutlocus). Using Ito calculus we infer that

$$dr_t^2 \leq 2r_t\, d\beta_t + r_t\Delta r(X_t)\, dt + dt, \tag{3.29}$$

where the last term comes from the quadratic variation. Together with the assumption that $\mathrm{Ric} \geq 0$, and hence $\Delta r \leq \frac{n-1}{r}$ by Laplace comparison, this yields

$$r_t^2 \leq 2\int_0^t r_s\, d\beta_s + nt\,. \tag{3.30}$$

Taking $t = \tau$ we see that the event $\{\tau \leq \delta\}$ implies

$$\int_0^\tau r_s\, d\beta_s \geq \frac{r^2 - n\tau}{2}. \tag{3.31}$$

By Levy's criterion there is a Brownian motion W such that

$$\int_0^\tau r_s\, d\beta_s = W_\sigma, \qquad \text{where} \qquad \sigma = \int_0^\tau r_s^2 ds \leq r^2\tau\,. \tag{3.32}$$

Hence $\{\tau \leq \delta\}$ implies

$$\max_{0 \leq s \leq r^2\delta} W_s \geq \frac{r^2 - n\delta}{2}. \tag{3.33}$$

We conclude that

$$\mathbb{P}_{x_0}[\tau \leq \delta] \leq \mathbb{P}_{x_0}\left[|W_1| \geq \frac{1}{r\delta^{1/2}}\frac{r^2 - n\delta}{2}\right] \leq e^{-\frac{r^2}{100\delta}}\,. \tag{3.34}$$

This proves the lemma. □

3.3. **Effective intersection estimate for two independent Brownian motions.** In this final section, we prove our effective intersection estimate.

Proof of Theorem 1.7. Since the two Brownian motions are independent, Theorem 1.1 yields

$$C^{-1}\mathbb{E}_{\tilde{x}_0}[\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X}))] \leq \mathbb{P}_{x_0,\tilde{x}_0}[N_{\varepsilon/2}(X) \cap N_{\varepsilon/2}(\tilde{X}) \neq \emptyset] \leq C\mathbb{E}_{\tilde{x}_0}[\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X}))]\,, \tag{3.35}$$

where we abbreviated $N_\varepsilon(X) = N_\varepsilon(X[0,\infty))$. We thus have the estimate the expected Martin capacity of the Wiener sausage from above and below.

Let us first prove the upper bound. To this end, first observe that for any Borel $A \subseteq M$ we can compute the expected volume via the formula

$$\mathbb{E}_{\tilde{x}_0}[\mathrm{Vol}(N_\varepsilon(\tilde{X}) \cap A)] = \int_A \mathbb{P}_{\tilde{x}_0}[\tilde{X}_t \in B(y,\varepsilon) \text{ for some } 0 < t < \infty\,]\, dV(y)\,. \tag{3.36}$$

Using again Theorem 1.1, as well as the equations (3.8) and (3.13) we can estimate this by

$$\begin{aligned}\mathbb{E}_{\tilde{x}_0}[\mathrm{Vol}(N_\varepsilon(\tilde{X})\cap A)] &\leq C\int_A \mathrm{Cap}_{K_{\tilde{x}_0}}(B(y,\varepsilon))\,dV(y)\\ &\leq C(d(\tilde{x}_0,A)-\varepsilon)_+^{2-n}\varepsilon^{n-2}\mathrm{Vol}(A)\,.\end{aligned}\tag{3.37}$$

On the other hand, for any Borel set $B\subseteq M$ we can estimate

$$\mathrm{Cap}_{K_{x_0}}(B)\leq d(x_0,B)^{2-n}\mathrm{Cap}_N(B)\leq Cd(x_0,B)^{2-n}\varepsilon^{-2}\mathrm{Vol}(N_\varepsilon(B))\,.\tag{3.38}$$

Combining the above, we infer that for any Borel set $A\subseteq M$ with $d(\tilde{x}_0,A)\geq 2\varepsilon$ we have the useful estimate

$$\mathbb{E}_{\tilde{x}_0}[\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X})\cap A)]\leq C\varepsilon^{n-4}d(x_0,A)^{2-n}d(\tilde{x}_0,A)^{2-n}\mathrm{Vol}(N_\varepsilon(A)),\tag{3.39}$$

which we will apply repeatedly in the following.

Set $r_0 := d(x_0,\tilde{x}_0)$ and let $E := B(x_0,r_0/2)\cup B(\tilde{x}_0,r_0/2)$. We decompose

$$N_\varepsilon(\tilde{X})\setminus E = N_\varepsilon(\tilde{X})\cap B(x_0,2r_0)\setminus E \cup \bigcup_{i=1}^{\infty} N_\varepsilon(\tilde{X})\cap B(x_0,2^{i+1}r_0)\setminus B(x_0,2^i r_0).\tag{3.40}$$

Hence, using (3.39) we can estimate

$$\begin{aligned}\mathbb{E}_{\tilde{x}_0}[\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X})\setminus E)] &\leq \sum_{i=0}^{\infty} C\varepsilon^{n-4}(2^i r_0)^{2-n}(2^i r_0)^{2-n}(2^i r_0)^n\\ &\leq C\varepsilon^{n-4}r_0^{4-n}.\end{aligned}\tag{3.41}$$

It remains to show similar bounds for the expected Martin capacity of $N_\varepsilon(\tilde{X})\cap E$. To this end, first observe that Theorem 1.1 yields

$$\begin{aligned}&\mathbb{E}_{\tilde{x}_0}[\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X})\cap B(x_0,2\varepsilon))]\\ &\qquad\leq \mathrm{Cap}_{K_{x_0}}(B(x_0,2\varepsilon))\mathbb{P}_{\tilde{x}_0}[\tilde{X}_t\in B(x_0,3\varepsilon)\text{ for some }0<t<\infty]\leq C\varepsilon^{n-2}r_0^{2-n}.\end{aligned}\tag{3.42}$$

Now, decomposing $B(x_0,r_0/2)\setminus B(x_0,2\varepsilon)$ into dyadic annuli with radii $2^i\varepsilon$, letting k be the smallest integer such that $2^k\varepsilon\geq r_0/2$, and using (3.39) we can estimate

$$\begin{aligned}\mathbb{E}_{\tilde{x}_0}[\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X})\cap B(x_0,r_0/2)\setminus B(x_0,2\varepsilon))] &\leq C\sum_{i=1}^{k}\varepsilon^{n-4}r_0^{2-n}(2^i\varepsilon)^{2-n}(2^i\varepsilon)^n\\ &\leq C\varepsilon^{n-4}r_0^{4-n}.\end{aligned}\tag{3.43}$$

Combining the above estimates, and repeating the same argument with x_0 and $\tilde{x}_0$ interchanged, we conclude that

$$\mathbb{E}_{\tilde{x}_0}[\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X})\cap E)]\leq C\varepsilon^{n-4}r_0^{4-n}.\tag{3.44}$$

This finishes the proof of upper bound.

For the lower bound, it suffices to show $\mathbb{E}_{\tilde{x}_0}[\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X})\cap A)] \geq C^{-1}\varepsilon^{n-4}r_0^{4-n}$ for some $A\subseteq M$. With this goal, a convenient choice is $A := B(x_0,4r_0)\setminus B(x_0,2r_0)$, as there is a positive $C=C(n,v)<\infty$ such that

$$C^{-1}r_0^n \leq \mathrm{Vol}(A) \quad\text{and}\quad C^{-1}r_0 \leq d(x_0,z), d(\tilde{x}_0,z) \leq Cr_0 \quad \text{for all } z\in A. \tag{3.45}$$

Inserting the uniform measure in the definition of $\mathrm{Cap}_{K_{x_0}}$ we see that

$$\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X})\cap A) \geq C^{-1}r_0^{2-n}\frac{\mathrm{Vol}(N_\varepsilon(\tilde{X})\cap A)^2}{\iint_{(N_\varepsilon(\tilde{X})\cap A)^2} d(x,y)^{2-n}\,dV(x)dV(y)}. \tag{3.46}$$

Arguing similarly as in the derivation of (3.37), but now using the lower bound of Theorem 1.1, we obtain

$$\mathbb{E}_{\tilde{x}_0}[\mathrm{Vol}(N_\varepsilon(\tilde{X})\cap A)] \geq C^{-1}r_0^2\varepsilon^{n-2}. \tag{3.47}$$

Together with the Cauchy-Schwarz inequality this yields

$$\mathbb{E}_{\tilde{x}_0}\left[\frac{\mathrm{Vol}(N_\varepsilon(\tilde{X})\cap A)^2}{\iint_{(N_\varepsilon(\tilde{X})\cap A)^2} d(x,y)^{2-n}\,dV(x)dV(y)}\right] \geq C^{-1}\frac{r_0^4\varepsilon^{2n-4}}{\mathbb{E}_{\tilde{x}_0}\left[\iint_{(N_\varepsilon(\tilde{X})\cap A)^2} d(x,y)^{2-n}\,dV(x)dV(y)\right]}. \tag{3.48}$$

To proceed, we rewrite the denominator as

$$\begin{aligned}\mathbb{E}_{\tilde{x}_0}&\left[\iint_{(N_\varepsilon(\tilde{X})\cap A)^2} d(x,y)^{2-n}\,dV(x)dV(y)\right]\\ &= \iint_{A\times A}\mathbb{P}_{\tilde{x}_0}[\tilde{X}\text{ hits } B(x,\varepsilon) \text{ and } B(y,\varepsilon)]d(x,y)^{2-n}\,dV(x)dV(y).\end{aligned} \tag{3.49}$$

By symmetry and the strong Markov property we can estimate

$$\begin{aligned}&\iint_{A\times A}\mathbb{P}_{\tilde{x}_0}[\tilde{X}\text{ hits } B(x,\varepsilon) \text{ and } B(y,\varepsilon)]d(x,y)^{2-n}\,dV(x)dV(y)\\ &\quad\leq 2\iint_{A\times A}\mathbb{P}_{\tilde{x}_0}[\tilde{X}\text{ hits } B(x,\varepsilon)]\sup_{x'\in B(x,\varepsilon)}\mathbb{P}_{x'}[X'\text{ hits } B(y,\varepsilon)]d(x,y)^{2-n}\,dV(x)dV(y)\,.\end{aligned} \tag{3.50}$$

Now, using again Theorem 1.1, for fixed x and x' we have

$$\begin{aligned}&\int_A \mathbb{P}_{x'}[X'\text{ hits } B(y,\varepsilon)]d(x,y)^{2-n}\,dV(y)\\ &\quad\leq C\int_{A\cap B(x,4\varepsilon)} d(x,y)^{2-n}dV(y) + C\int_{A\setminus B(x,4\varepsilon)}\varepsilon^{n-2}d(x',y)^{2-n}d(x,y)^{2-n}\,dV(y) \leq C\varepsilon^2\,.\end{aligned} \tag{3.51}$$

Combining the above formulas we infer that

$$\mathbb{E}_{\tilde{x}_0}\left[\iint_{(N_\varepsilon(\tilde{X})\cap A)^2} d(x,y)^{2-n}\,dV(x)dV(y)\right] \leq C\varepsilon^2\int_A \mathbb{P}_{\tilde{x}_0}[\tilde{X}\text{ hits } B(x,\varepsilon)]\,dV(x) \leq Cr_0^2\varepsilon^n\,, \tag{3.52}$$

and thus conclude that

$$\mathbb{E}_{\tilde{x}_0}[\mathrm{Cap}_{K_{x_0}}(N_\varepsilon(\tilde{X})\cap A)] \geq C^{-1}r_0^{4-n}\varepsilon^{n-4}\,. \tag{3.53}$$

This finishes the proof of the theorem. ☐

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