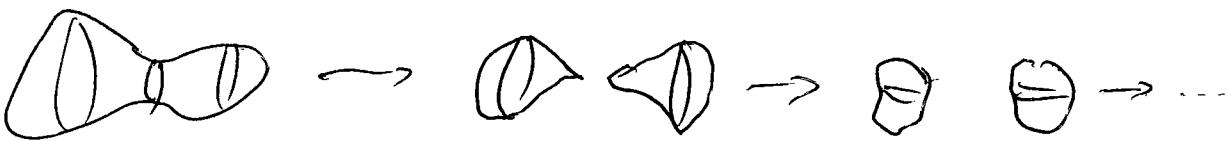


Weak/generalized solutions of the MCF

Goal: want a generalized no.

Idea:



Approaches:

- 1) Level set flow (~~comparison principle~~ avoidance principle)
- 2) Brakke flow (geometric measure theory)
- 3) flow with surgery (cut along necks & glue in caps)

(1) Level set flow
~~(avoidance principle)~~:

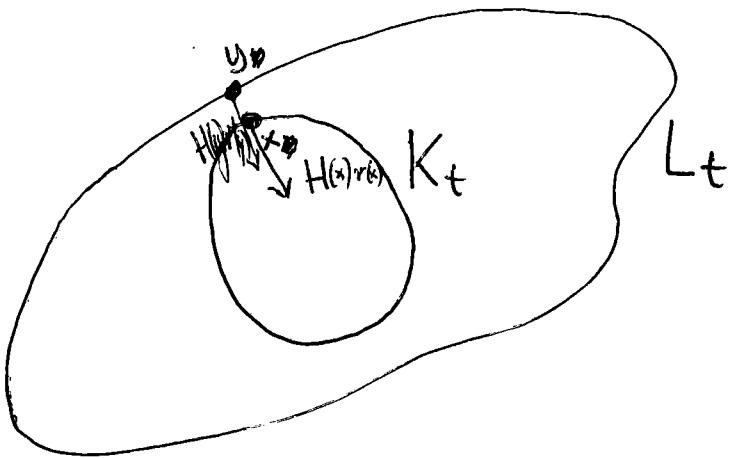
recall If $\{K_t\}_{t \in [t_0, t_1]}$, $\{L_t\}_{t \in [t_0, t_1]}$ are smooth MCFs, and at least one of them is compact, then

$$K_{t_0} \cap L_{t_0} = \emptyset \Rightarrow K_t \cap L_t = \emptyset \quad \forall t \geq t_0.$$

or equivalently

$t \mapsto \text{dist}(K_t, L_t)$ is nondecreasing.

(2)



$$dist(K_t, L_t)$$

$$\begin{aligned} & \cancel{\text{def}} \|x - y\| \\ &= \inf_{\substack{x \in K_t \\ y \in L_t}} \|x - y\| \end{aligned}$$

$$\frac{d}{dt} dist(K_t, L_t) \geq H(x, t) - H(y, t) \geq 0.$$

Let $K \subset \mathbb{R}^n$ be a closed set.

Def A family $\{K_t\}_{t \in [0, T]}$ of closed sets is a set theoretic subsolution of the MCF starting at $K_0 = K$ if

$\forall \{L_t\}_{t \in [t_0, t_1]}$ smooth cpt MCF

$$K_{t_0} \cap L_{t_0} = \emptyset \Rightarrow K_t \cap L_t = \emptyset \quad \forall t \geq t_0.$$

Rmk By translation invariance of the smooth cpt MCF $\{L_t\}$ this is equivalent to $t \mapsto dist(K_t, L_t)$ is nondecreasing.

Def Let $K \subset \mathbb{R}^N$ be a closed set. (3)

The Level set flow of K is the maximal set theoretic subsolution $\{K_t\}_{t \in [0, T]}$ with $K_0 = K$.

Maximal means:

If $\{K'_t\}$ is any other set theoretic subsolution,
then $K'_t \subseteq K_t \quad \forall t \in [0, T]$.

Prop For any closed set $K \subset \mathbb{R}^N$ there exists a unique solution $\{K_t\}_{t \in [0, T]}$ of the Level set flow with $K_0 = K$.

Proof

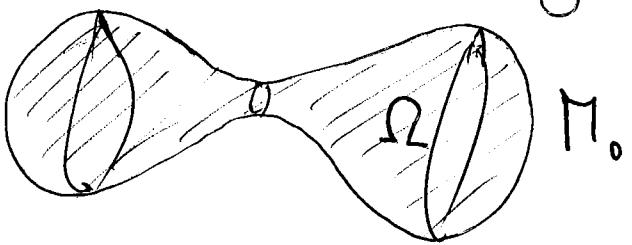
$$K_t = \overline{\bigcup K'_t}$$

$\{K'_t\}_{t \in [0, T]}$ is a set theoretic subsolution
of the MCF starting at K . □

(4)

Description in terms of PDEs

Assume $M_0^n \subset \mathbb{R}^{n+1}$ is a closed mean convex hypersurface. Let Ω be the domain bounded by M_0 .



Idea: $\{M_t\}_{t \in [0, \infty)}$ the evolved flow of M_0

$\forall x \in \Omega \exists ! t = 0$ st. $x \in M_t$

this defines the time of arrival function

$$u: \Omega \rightarrow \mathbb{R}$$

$$u(x) = t \Leftrightarrow x \in M_t$$

Q: Which eqn does u solve?

(5)

$$\text{Diagram: A blob-like domain } \Omega \text{ with boundary } \partial\Omega. \text{ Inside, } u(x) = t.$$
$$-\operatorname{div} \left(\frac{Du}{|Du|} \right) = \frac{1}{|Du|}$$

mean curvature speed

Thus we want to solve the
Dirichlet problem

$$(*) \quad \left\{ \begin{array}{l} \operatorname{div} \left(\frac{Du}{|Du|} \right) + \frac{1}{|Du|} = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{array} \right.$$

in the viscosity sense (see later)

(6)

The problem (*) is degenerate elliptic

(it is degenerate at points where $|Du| = 0$)

Elliptic regularization

$$(*)_\varepsilon \quad \left\{ \begin{array}{l} \operatorname{div} \left(\frac{Du_\varepsilon}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} \right) + \frac{1}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} = 0 \text{ in } \Omega \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \end{array} \right.$$


Thm Let $\Omega \subset \mathbb{R}^N$ be a compact mean convex domain.

Then for every $\varepsilon > 0$, the problem $(*)_\varepsilon$ has a unique smooth solution u_ε .

Moreover

$$\sup_{\varepsilon \in (0,1)} \sup_{x \in \Omega} |Du_\varepsilon|(x) \leq C.$$

$$\text{Proof o)} \quad \operatorname{div} \left(\frac{Du_\varepsilon}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} \right) + \frac{1}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} = 0 \quad (7)$$

$$\Leftrightarrow \underbrace{\left(\delta_{ij} - \frac{D_i u_\varepsilon D_j u_\varepsilon}{(\varepsilon^2 + |Du_\varepsilon|^2)} \right)}_{=: \alpha_{ij}(Du_\varepsilon)} D_i D_j u_\varepsilon + 1 = 0$$

If u_ε and \hat{u}_ε are two solutions, then at an interior min. of $v_\varepsilon = u_\varepsilon - \hat{u}_\varepsilon$ we have $Du_\varepsilon = D\hat{u}_\varepsilon$ and thus $\alpha_{ij}(Du_\varepsilon) D_i D_j v_\varepsilon = 0 \Rightarrow v_\varepsilon \geq 0$.

Changing $u_\varepsilon \leftrightarrow \hat{u}_\varepsilon$ gives $v_\varepsilon \leq 0 \Rightarrow \text{uniqueness}$.

•) to prove existence we use the continuity method.

Fix $\varepsilon \in (0, 1)$. Consider

$$(*)_{\varepsilon, \gamma} \left\{ \begin{array}{l} \operatorname{div} \left(\frac{Du_{\varepsilon, \gamma}}{\sqrt{\varepsilon^2 + |Du_{\varepsilon, \gamma}|^2}} \right) + \frac{\gamma}{\sqrt{\varepsilon^2 + |Du_{\varepsilon, \gamma}|^2}} = 0 \text{ in } \Omega \\ u_{\varepsilon, \gamma} = 0 \quad \text{on } \partial\Omega \end{array} \right.$$

Let $I := \{ \gamma \in [0, 1] \mid \text{problem } (*)_{\varepsilon, \gamma} \text{ has a solution} \}$

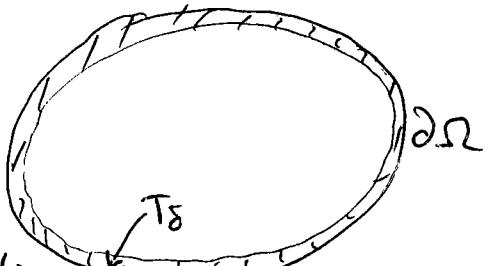
Obviously $0 \in I$. ($u \equiv 0$ is a solution of $(*)_{\varepsilon, 0}$)

To show $I \in I$, it is enough to show that (8)
 I is open & closed.

Claim: $\sup_{\partial\Omega} |Du_{\varepsilon, \kappa}| \leq C(\varepsilon, \Omega)$ indep. of κ .

Proof Let $r(x) := d(x, \partial\Omega)$.

$$\delta > 0, C < \infty.$$



Will show: $v := Cr$ is a supersolution on T_δ (for suitable C, δ).

$$\operatorname{div}\left(\frac{Dv}{\sqrt{\varepsilon^2 + |Dv|^2}}\right) + \frac{\gamma_\kappa}{\sqrt{\varepsilon^2 + |Dv|^2}} \leq \underbrace{\frac{C \Delta r}{\sqrt{\varepsilon^2 + C^2}}}_{\text{Since } H_0 = \min_{\partial\Omega} H > 0} + \underbrace{\frac{1}{\sqrt{\varepsilon^2 + C^2}}}$$

Since $H_0 = \min_{\partial\Omega} H > 0$, $\exists \delta > 0$ st

r is smooth on T_δ and $-\Delta r \geq \frac{1}{2} H_0$ on T_δ .

Choose C large enough st ≤ 0 and

$v \geq u_{\varepsilon, \kappa}$ on $\overline{T_\delta}$ (note that $\sup u_{\varepsilon, \kappa} \leq C(\varepsilon)$)
 Exer

$$\Rightarrow u_{\varepsilon, \kappa} \leq v \text{ on } \overline{T_\delta} \Rightarrow \text{Claim } \square$$

Now consider

$$\Omega \subset \mathbb{R}^n$$



$$M^{\varepsilon, \kappa} = \text{graph}\left(\frac{u_{\varepsilon, \kappa}}{\varepsilon}\right) \subset \Omega \times \mathbb{R}_+$$

$$(*)_{\varepsilon, \kappa} \Leftrightarrow \underbrace{H + \frac{\kappa}{\varepsilon}}_{\text{mean curvature of } M^{\varepsilon, \kappa}} < \langle e_{n+1}, \gamma \rangle = 0$$

Indeed:

$$\gamma = \frac{\ell}{\sqrt{\ell^2}}$$

Indeed: $F(x) \mapsto (x, \frac{u(x)}{\varepsilon})$ per. of M

$$\text{tangent vectors } \partial_i F = \left(0, \dots, \underset{i}{1}, 0, \dots, 0, \frac{\partial_i u(x)}{\varepsilon} \right)$$

$$\text{normal vector } \gamma = \begin{pmatrix} D u_\varepsilon \\ -1 \end{pmatrix} \frac{1}{\sqrt{1 + |Du_\varepsilon|^2}}$$

$$\frac{\kappa}{\varepsilon} \langle e_{n+1}, \gamma \rangle = -\frac{\kappa}{\varepsilon} \frac{1}{\sqrt{1 + |Du|^2/\varepsilon^2}} = \frac{-\kappa}{\sqrt{\varepsilon^2 + |Du|^2}}$$

$$H = -\text{div}_{\Gamma} (\gamma) = -\text{div}_{\Gamma} \left(\frac{(Du, -\varepsilon)}{\sqrt{\varepsilon^2 + |Du|^2}} \right)$$

$$= -\text{div}_{\mathbb{R}^n} \left(\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \right).$$

Compute on $H^{\varepsilon, \kappa}$:

(10)

$$\Delta \langle e_{N+1}, r \rangle = \langle e_{N+1}, \nabla H \rangle - |A|^2 \langle e_{N+1}, r \rangle$$

(this is just $\partial_t H = \Delta H + |A|^2 H$ in disguise)

$$\text{insert } \nabla H = -\frac{\kappa}{\varepsilon} \nabla \langle e_{N+1}, r \rangle$$

max-princ. $\Rightarrow -\langle e_{N+1}, r \rangle$ attains its
maximum at $\partial\Omega$.

$$\Rightarrow -\langle e_{N+1}, r \rangle = \frac{+\varepsilon}{\sqrt{\varepsilon^2 + |Du|^2}}$$

~~$\geq C(\varepsilon, \Omega)$~~
 $C(\varepsilon, \Omega) > 0$
indep. of κ by

$$\geq C(\varepsilon, \Omega) > 0$$

↑ indep. of κ by Claim

$$\Rightarrow |Du_{\varepsilon, \kappa}| \leq C(\varepsilon, \Omega).$$



(11)

DeGiorgi-Nash-Moser/Schauder \Rightarrow

κ_K -indep. estimates for higher derivatives of $u_{\varepsilon, \kappa}$.

I closed: $k_m \in I$, $k_m \rightarrow \kappa$

est. $u_{\varepsilon, k_m} \rightarrow u_{\varepsilon, \kappa}$ solutions of $(*)_{\varepsilon, \kappa}$.

I open: $M_\kappa: C_0^{2,\alpha}(\Omega) \rightarrow C^\alpha(\Omega)$

$$M_\kappa(u) = \operatorname{div} \left(\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \right) + \frac{\kappa}{\sqrt{\varepsilon^2 + |Du|^2}}$$

Linearization at $u = u_{\varepsilon, 0}$:

$$\begin{aligned} L_\kappa(v) &= \operatorname{div} \left(\frac{Dv}{\sqrt{\varepsilon^2 + |Du|^2}} - \frac{\langle Du, Dv \rangle Du}{(\varepsilon^2 + |Du|^2)^{3/2}} \right) \\ &\quad - \frac{\kappa \langle Du, Dv \rangle}{(\varepsilon^2 + |Du|^2)^{3/2}} \end{aligned}$$

At pos max of v: $L_\kappa(v) \leq 0$

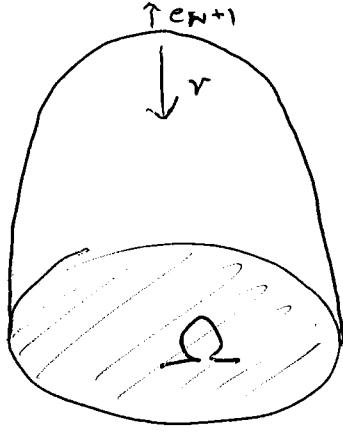
At $L_\kappa(v) = 0 \Rightarrow v = 0$ max.princ: $L_\kappa(v) = 0 \underset{v=0 \text{ on } \partial\Omega}{\Rightarrow} v = 0$

$\Rightarrow L_\kappa: C_0^{2,\alpha}(\Omega) \rightarrow C^\alpha(\Omega)$ invertible

inv. fn. thm $\Rightarrow M: [0,1] \times C_0^{2,\alpha}(\Omega) \rightarrow [0,1] \times C^\alpha(\Omega)$, $M(\chi u) = f u$, invertible

$\Omega \subset \mathbb{R}^N$ mean convex domain

$$(*) \begin{cases} |\nabla u| \operatorname{div} \left(\frac{\nabla u}{|\nabla u|} \right) + 1 = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



$$(*)_\varepsilon \begin{cases} \operatorname{div} \left(\frac{\nabla u_\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} \right) + \frac{1}{\sqrt{\varepsilon^2 + |\nabla u_\varepsilon|^2}} = 0 & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

recall: i) For every $\varepsilon > 0$ the problem $(*)_\varepsilon$ has a unique solution u_ε . Moreover

$$\sup_{\varepsilon \in (0,1)} \sup_{x \in \Sigma} |\nabla u_\varepsilon|(x) \leq C.$$

ii) The geometric meaning of $(*)_\varepsilon$ is that

$$M^\varepsilon := \operatorname{graph} \left(\frac{u_\varepsilon}{\varepsilon} \right) \subset \mathbb{R}^N \times \mathbb{R} \text{ satisfies}$$

$$H + \frac{1}{\varepsilon} \langle e_{N+1}, r \rangle = 0$$

i.e. $M_t^\varepsilon = \operatorname{graph} \left(\frac{u_\varepsilon - t}{\varepsilon} \right)$ is a selfsimilarly

translating solution of the MCF in $\mathbb{R}^N \times \mathbb{R}$

that moves in ~~the~~ direction $-e_{N+1}$

with speed $\frac{1}{\varepsilon}$.

Viscosity solutions of (*)

(2)

$$\left(\delta_{ij} - \frac{D_i u D_j u}{|Du|^2} \right) D_{ij} u - 1 = 0$$



Def $u: \Omega \rightarrow \mathbb{R}$ continuous is a viscosity subsolution
 if $\forall x_0 \in \Omega$ ~~for some~~ tangent to u from above at x_0
 (i.e. $\varphi \geq u$, $\varphi(x_0) = u(x_0)$)

we have

$$\text{either } D\varphi \neq 0 \text{ and } -\left(\delta_{ij} - \frac{D_i \varphi D_j \varphi}{|D\varphi|^2}\right) D_{ij} \varphi + 1 \leq 0 \text{ at } x_0$$

$$\text{or } D\varphi = 0 \text{ and } -(\delta_{ij} - v_i v_j) D_{ij} \varphi + 1 \leq 0 \text{ at } x_0$$

for some ~~unit~~ unit vector v , $|v| \leq 1$.

viscosity supersolution: tang. from below, \geq

visc. solution = visc. sub & super-solution.

Prop For a ~~sub~~sequence $\varepsilon_k \rightarrow 0$ the functions

u_{ε_k} converge uniformly to a Lipschitz fn u which is a viscosity solution of (*).

Proof

$$\sup_{\varepsilon \in (0,1)} \sup_{x \in \Omega} |Du_\varepsilon|(x) \leq C$$

$\Rightarrow \exists \varepsilon_k \rightarrow 0$ s.t. $u_{\varepsilon_k} \rightarrow u$ uniformly, u Lipschitz.

will show: u visc. subsolution (super sol. similar)

Let φ be tangent to u from above at x_0 .

Without changing the first & second derivative of φ we can assume

$$\varphi > u \text{ except at } x_0$$

$$\varphi > u + 1 \text{ outside } B_r(x_0) = B$$

\Rightarrow for k large enough $\exists x_k \in B$, $\varphi_k = \varphi + \delta_k$ s.t.

φ_k is tangent to u_{ε_k} from above at x_k .

Moreover $x_k \rightarrow x$, $\delta_k \rightarrow 0$.

Note that $-(\delta_{ij} - \frac{D_i \varphi_k D_j \varphi_k}{\varepsilon_k^2 + |D\varphi_k|^2}) D_{ij} \varphi_k + 1 \leq 0$ at x_k .

By since $\varphi_k(x_0) = u_\varepsilon(x_0)$, $D\varphi_k(x_0) = Du_\varepsilon(x_0)$, $D^2\varphi_k(x_0) \geq D^2u_\varepsilon(x_0)$

If $D\varphi(x_0) \neq 0$, then $D\varphi_{k\epsilon}(x_k)$ bold away from 0
for large

(4)

$$\Rightarrow -(\delta_{ij} - \frac{D_i\varphi D_j\varphi}{|D\varphi|^2})D_{ij}\varphi + 1 \leq 0 \text{ at } x_0$$

If $D\varphi(x_0) = 0$, then $\frac{D\varphi_{k\epsilon}}{\sqrt{|D\varphi_{k\epsilon}|^2 + \epsilon_k^2}} \rightarrow v, |v| \leq 1$
along a subsequence

$$\Rightarrow -(\delta_{ij} - v_i v_j)D_{ij}\varphi + 1 \leq 0 \text{ at } x.$$

(1)

Cor $v(x, t) := u(x) - t$

is a viscosity solution of the level set flow eqn

$$\partial_t v = \left(\delta_{ij} - D_i v D_j v / |Dv|^2 \right) D_{ij} v \quad (**)$$

Consequence (**)

The geom. meaning of (**) is that all level sets

$$\Gamma_t^a = \{x \in \mathbb{R}^N : v(x, t) = a\}$$

move by MCF.

Cg

~~Not~~ Clearly Γ_t^a is a set theoretic subsolution

Note: v visc. sol of (**) $\Rightarrow \{\Gamma_t^a\}_{t \geq 0}$ is a
set theoretic subsol.
of the MCF.



(from def of visc. sol.)

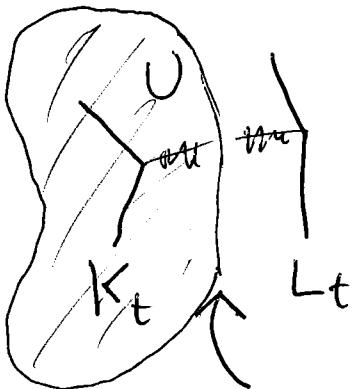
(6)

Lemma (Ilmanen)

$\{K_t\}_{t \geq 0}, \{L_t\}_{t \geq 0}$ set-theoretic subs. of
the MCF ; K_0 compact ;

$$\underline{K_0 \cap L_0 = \emptyset} \Rightarrow \cancel{\text{dist}(K_t, L_t) \text{ nondecr}} \\ \Rightarrow t \mapsto \text{dist}(K_t, L_t) \text{ nondecreasing.}$$

Idea



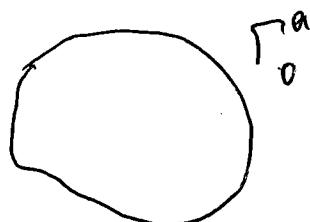
$\exists C^{1,1}$ -hypersurface $Q \subset \partial U$

$$K_t \subseteq U, \quad L_t \subseteq \mathbb{R}^N \setminus \bar{U}$$

$$\text{dist}(K_t, Q) + \text{dist}(Q, L_t) = \text{dist}(K_t, L_t)$$

Cor The notions of set theoretic solution and viscosity solution are equivalent.

Proof already know Γ_t^a is a set theoretic subsolution.



$$K_0 \subseteq \Gamma_0^a \Rightarrow K_t \subseteq \Gamma_t^a$$

Lemma

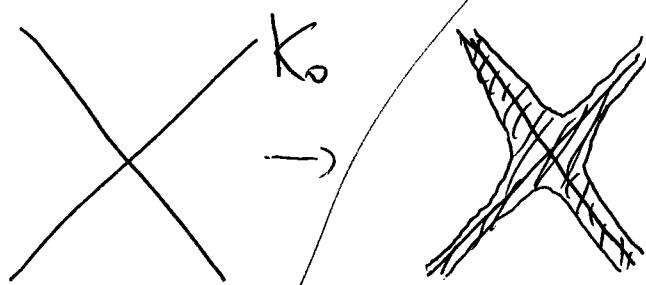
~~since other~~ (otherwise would run into other based)

$\Rightarrow \Gamma_t^a$ is the maximal set theoretic subsolution

Level set flow:

(+) can flow any closed set

(-) fattening, doesn't depend continuously on initial data, ~~thus~~ ^{not preserved} under limits



K_t develops interior
(cannot happen in mean convex case)

