

(1)

The local regularity theorem

$\mathcal{M} = \{M_t\}$ mean curvature flow.

Recall that $\rho_{(x_0, t_0)}(x, t) = \frac{1}{(4\pi(t_0-t))^{n/2}} e^{-\frac{|x-x_0|^2}{4(t_0-t)}}$ ($t < t_0$)

satisfies $\left(\frac{d}{dt} + \Delta_{M_t} - H^2\right) \rho_{(x_0, t_0)} = -|H - \frac{\langle x-x_0, r \rangle}{2(t_0-t)}|^2 \rho_{(x_0, t_0)}$

and thus $\frac{d}{dt} \int_{M_t} \rho_{(x_0, t_0)} d\mu = - \int_{M_t} |H - \frac{\langle x-x_0, r \rangle}{2(t_0-t)}|^2 \rho_{(x_0, t_0)} d\mu$

(Huisken's monotonicity formula)

Localized version: If $\{M_t\}$ is only defined locally, say in $B(x_0, \sqrt{n}\rho) \times (t_0 - \rho^2, t_0)$, then we can use the cutoff function

$$\rho_{(x_0, t_0)}^\rho(x, t) = \left(1 - \frac{|x-x_0|^2 + 2n(t-t_0)}{\rho^2}\right)_+^3$$

Since $\left(\frac{d}{dt} - \Delta_{M_t}\right) \rho_{(x_0, t_0)}^\rho \leq 0$, we still get the monotonicity inequality

$$\frac{d}{dt} \int_{M_t} \rho_{(x_0, t_0)} \rho_{(x_0, t_0)}^\rho d\mu \leq - \int_{M_t} |H - \frac{\langle x-x_0, r \rangle}{2(t_0-t)}|^2 \rho_{(x_0, t_0)} \rho_{(x_0, t_0)}^\rho d\mu$$

Indeed:

(2)

$$\begin{aligned} \frac{d}{dt} \int_{M_t} \rho \varphi d\mu &= \int_{M_t} \left(\frac{d}{dt} \rho \varphi + \rho \frac{d}{dt} \varphi - \rho |\nabla H|^2 \right) d\mu \\ &= \int_{M_t} \left(\underbrace{\left(\frac{d}{dt} + \Delta_{M_t} - H^2 \right) \rho \cdot \varphi}_{\leq 0} + \rho \cdot \underbrace{\left(\frac{d}{dt} - \Delta_{M_t} \right) \varphi}_{\leq 0} \right) d\mu \\ &= -|H - \frac{\langle x - x_0, r \rangle}{2(t_0 - t)}|^2 \rho \end{aligned}$$

□

The monotone quantity

$$\textcircled{H}^r(M, (x_0, t_0), r) = \int_{M_{t_0-r^2}} \rho_{(x_0, t_0)} \varphi_{(x_0, t_0)}^r d\mu$$

is called the Gaussian density ratio.

Note: $\textcircled{H}^\infty(M, (x_0, t_0), r)$

Note: If $(x_0, t_0) \in M$ (i.e. $x_0 \in M_{t_0}$) then

$\textcircled{H}^\infty(M, (x_0, t_0), r) \geq 1$ and (with multiplicity one)

$\textcircled{H}^\infty(M, (x_0, t_0), r) = 1 \quad \forall r \in (0, \infty) \iff M$ is a flat plane.

(3)

Thm (Local regularity thm)

There exist universal constants $\varepsilon > 0, C < \infty$ (depending only on the dimension) with the following significance. If M is a smooth MCF in a parabolic ball $P(x_0, 4np)$ with

$$\sup_{X \in P(x_0, r)} \Theta^p(M, X, r) < 1 + \varepsilon$$

for some $r \in (0, \varepsilon)$ then

$$\sup_{P(x_0, r/2)} |A| \leq Cr^{-1}$$

Notation: $X_0 = (x_0, t_0)$.

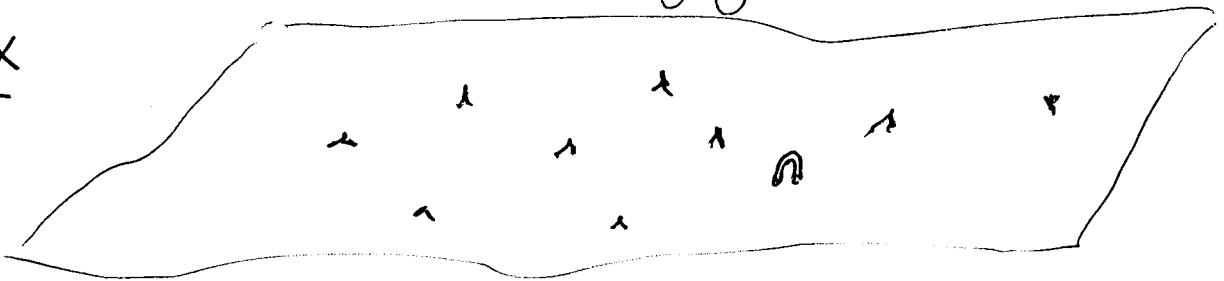
$$P(x_0, r) = B(x_0, r) \times [t_0 - r^2, t_0]$$

Rmk If $\Theta < 1 + \varepsilon/2$ holds at some point and some scale,

then $\Theta < 1 + \varepsilon$ holds at all nearby points and somewhat smaller scales. (Exer!)

(4)

- Rmk) $\text{H} < 1 + \varepsilon$ means that the flow
is "weakly" (in a measure theoretic sense)
close to a plane.
•) $|A|$ small \Leftrightarrow strongly close to a plane.

Ex

weakly but not strongly close to a plane.

The fact that we have an open (MCF)
and a monotone quantity (Huisker)
rules out examples like this!

We will now discuss the local regularity theorem for the mean curvature flow, which gives definite curvature bounds in a neighborhood of definite size, provided the Gaussian density ratio is close to one.

Since time scales like distance squared, the natural neighborhoods to consider are parabolic balls $P(x_0, t_0, r) = B(x_0, r) \times (t_0 - r^2, t_0]$.

Theorem 2.14 (Local regularity theorem [Bra78, Whi05]). *There exist universal constants $\varepsilon > 0$ and $C < \infty$ with the following property. If \mathcal{M} is a smooth mean curvature flow in a parabolic ball $P(X_0, 4n\rho)$ with*

$$(2.15) \quad \sup_{X \in P(X_0, r)} \Theta^\rho(\mathcal{M}, X, r) < 1 + \varepsilon$$

for some $r \in (0, \rho)$, then

$$(2.16) \quad \sup_{P(X_0, r/2)} |A| \leq Cr^{-1}.$$

Remark 2.17. If $\Theta < 1 + \frac{\varepsilon}{2}$ holds at some point and some scale, then $\Theta < 1 + \varepsilon$ holds at all nearby points and somewhat smaller scales.

Proof of Theorem 2.14. Suppose the assertion fails. Then there exist a sequence of smooth flows \mathcal{M}^j in $P(0, 4n\rho_j)$, for some $\rho_j > 1$, with

$$(2.18) \quad \sup_{X \in P(0, 1)} \Theta^{\rho_j}(\mathcal{M}^j, X, 1) < 1 + j^{-1},$$

but such that there are points $X_j \in P(0, 1/2)$ with $|A|(X_j) > j$.

By point selection, we can find $Y_j \in P(0, 3/4)$ with $Q_j = |A|(Y_j) > j$ such that

$$(2.19) \quad \sup_{P(Y_j, j/10Q_j)} |A| \leq 2Q_j.$$

Let us explain how the point selection works: Fix j . If $Y_j^0 = X_j$ already satisfies (2.19) with $Q_j^0 = |A|(Y_j^0)$, we are done. Otherwise, there is a point $Y_j^1 \in P(Y_j^0, j/10Q_j^0)$ with $Q_j^1 = |A|(Y_j^1) > 2Q_j^0$. If Y_j^1 satisfies (2.19), we are done. Otherwise, there is a point $Y_j^2 \in P(Y_j^1, j/10Q_j^1)$ with $Q_j^2 = |A|(Y_j^2) > 2Q_j^1$, etc. Note that $\frac{1}{2} + \frac{j}{10Q_j^0}(1 + \frac{1}{2} + \frac{1}{4} + \dots) < \frac{3}{4}$. By smoothness, the iteration terminates after a finite number of steps, and the last point of the iteration lies in $P(0, 3/4)$ and satisfies (2.19).

Continuing the proof of the theorem, let $\hat{\mathcal{M}}^j$ be the flows obtained by shifting Y_j to the origin and parabolically rescaling by $Q_j = |A|(Y_j) \rightarrow \infty$. Since the rescaled flow satisfies $|A|(0) = 1$ and $\sup_{P(0, j/10)} |A| \leq 2$, we can pass smoothly to a nonflat global limit. On the other hand, by the rigidity case of (2.12), and since

$$(2.20) \quad \Theta^{\hat{\rho}_j}(\hat{\mathcal{M}}^j, 0, Q_j) < 1 + j^{-1},$$

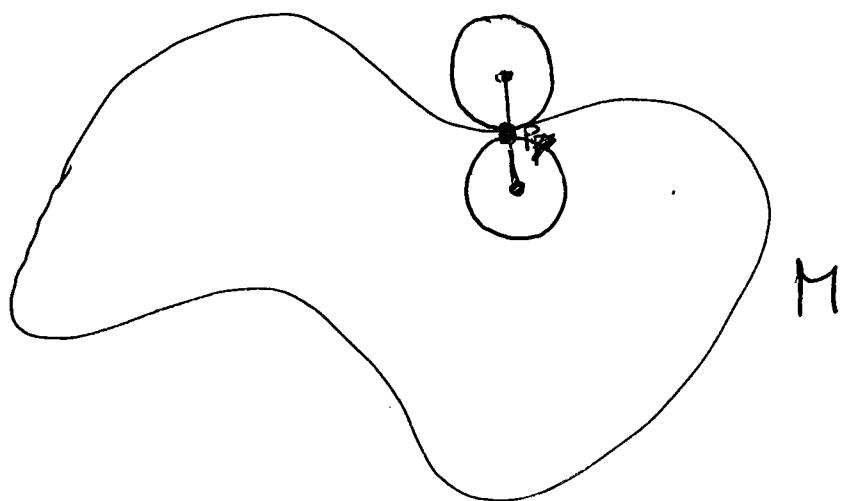
where $\hat{\rho}_j = Q_j \rho_j \rightarrow \infty$, the limit is a flat plane; a contradiction. \square

Noncollapsing (quantitative embeddedness)

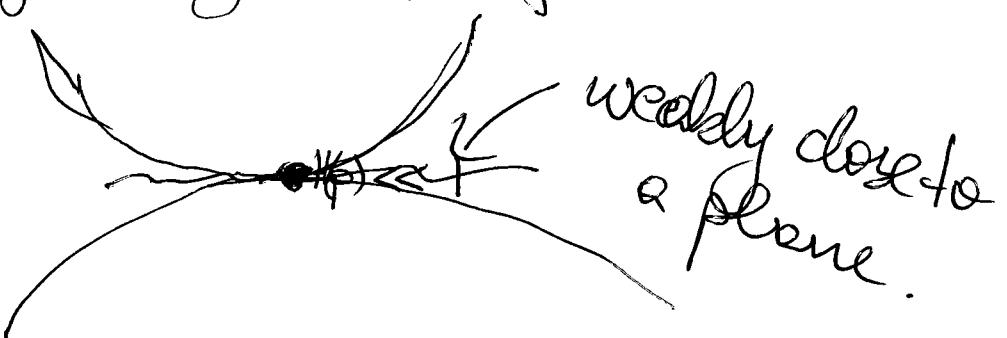
①

Def A closed embedded mean convex hypersurface $M^n \subset \mathbb{R}^{n+1}$ is called α -noncollapsed

if each $p \in M$ admits interior & exterior balls tangent at p of radius $\frac{\alpha}{\sqrt{H(p)}}$



Rmk Can be combined ~~nicely~~ well with the local regularity theorem of next lecture.



(2)

Thm (Andrews)

α -noncollapsing is preserved under MCF,

i.e. $M_0 \text{ } \alpha\text{-noncollapsed} \Rightarrow M_t \text{ } \alpha\text{-noncollapsed}$

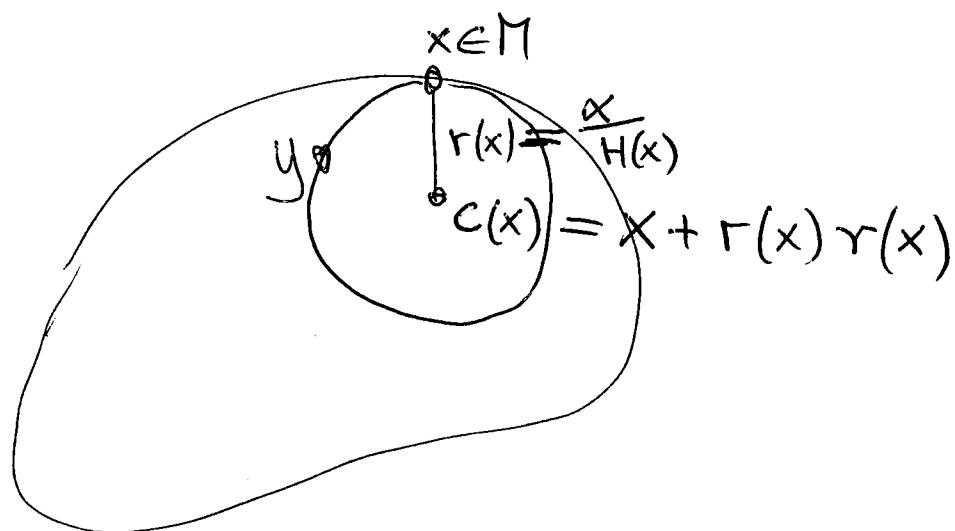


(same α !)

Rmk By compactness, M_0 is α -noncollapsed
for some $\alpha > 0$.

1st step of proof:

express interior/exterior ball condition
in terms of certain inequalities:



(3)

interior ball:

$$\|y - c(x)\|^2 \geq r(x)^2 \quad \forall y \in M.$$

~~top~~ $\Rightarrow \|y - x - r(x)v(x)\|^2 \geq r(x)^2 \quad \forall y \in M$

$$\Leftrightarrow \|y - x\|^2 - 2\underbrace{r(x)}_{=H(x)} \langle y - x, v(x) \rangle + \cancel{r(x)^2} \geq \cancel{r(x)^2} 0. \quad \forall y \in M$$

$$\Leftrightarrow \frac{2 \langle y - x, v(x) \rangle}{\|y - x\|^2} \leq \frac{H(x)}{\alpha} \quad \forall y \in M$$

Define $Z^*(x, t) := \sup_{y \neq x} \frac{2 \langle F(y, t) - F(x, t), v(x, t) \rangle}{\|F(y, t) - F(x, t)\|^2}$

Proving interior α -noncollapsing amounts to

Showing $Z^*(x, t) \leq H(x, t)/\alpha$ at $t = 0$

$$\Rightarrow Z^*(x, t) \leq H(x, t)/\alpha \quad \forall t \geq 0.$$

Similarly, $Z_*(x, t) := \inf_{y \neq x} \frac{2 \langle F(y, t) - F(x, t), v(x, t) \rangle}{\|F(y, t) - F(x, t)\|^2} \geq -\frac{H(x, t)}{\alpha}$

ROBERT HASLHOFER

$c(x) = x + r(x)\nu(x)$. The condition that this is indeed an interior ball is equivalent to the inequality

$$(3.5) \quad \|y - c(x)\|^2 \geq r(x)^2 \quad \text{for all } y \in M.$$

Observing $\|y - c(x)\|^2 = \|y - x\|^2 - 2r(x)\langle y - x, \nu(x) \rangle + r(x)^2$ and inserting $r(x) = \frac{\alpha}{H(x)}$ the inequality (3.5) can be rewritten as

$$(3.6) \quad \frac{2\langle y - x, \nu(x) \rangle}{\|y - x\|^2} \leq \frac{H(x)}{\alpha} \quad \text{for all } y \in M.$$

Now given a mean convex flow $M_t = X(M, t)$ of closed embedded hypersurfaces, we consider the quantity

$$(3.7) \quad Z^*(x, t) = \sup_{y \neq x} \frac{2\langle X(y, t) - X(x, t), \nu(x, t) \rangle}{\|X(y, t) - X(x, t)\|^2}.$$

Proving interior noncollapsing amounts to showing that if

$$(3.8) \quad Z^*(x, t) \leq \frac{H(x, t)}{\alpha}$$

holds at $t = 0$, then this holds for all t . Similarly, exterior noncollapsing amounts to proving the inequality

$$(3.9) \quad Z_*(x, t) = \inf_{y \neq x} \frac{2\langle X(y, t) - X(x, t), \nu(x, t) \rangle}{\|X(y, t) - X(x, t)\|^2} \geq -\frac{H(x, t)}{\alpha}.$$

That the inequalities (3.8) and (3.9) are indeed preserved under mean curvature flow is a quick consequence of the following theorem.

Theorem 3.10 (Andrews-Langford-McCoy [ALM13]). *Let M_t be a mean curvature flow of closed embedded mean convex hypersurfaces, and define Z_* and Z^* as in (3.7) and (3.9). Then*

$$(3.11) \quad \partial_t Z_* \geq \Delta Z_* + |A|^2 Z_* \quad \partial_t Z^* \leq \Delta Z^* + |A|^2 Z^*$$

in the viscosity sense.

Proof of Theorem 3.3 (using Theorem 3.10). We start by computing

$$(3.12) \quad (\partial_t - \Delta) \frac{Z_*}{H} = \frac{(\partial_t - \Delta) Z_*}{H} - \frac{Z_*(\partial_t - \Delta) H}{H^2} + 2\langle \nabla \log H, \nabla \frac{Z_*}{H} \rangle.$$

Thus, using Proposition 1.8 and Theorem 3.10, we obtain

$$(3.13) \quad \partial_t \frac{Z_*}{H} \geq \Delta \frac{Z_*}{H} + 2\langle \nabla \log H, \nabla \frac{Z_*}{H} \rangle.$$

By the maximum principle, the minimum of $\frac{Z^*}{H}$ is nondecreasing in time. In particular, if the inequality $\frac{Z^*}{H} \geq -\frac{1}{\alpha}$ holds at $t = 0$, then this inequality holds for all t . Arguing similarly we obtain that

$$(3.14) \quad \partial_t \frac{Z^*}{H} \leq \Delta \frac{Z^*}{H} + 2\langle \nabla \log H, \nabla \frac{Z^*}{H} \rangle,$$

and thus that the inequality $\frac{Z^*}{H} \leq \frac{1}{\alpha}$ is also preserved along the flow. \square

It remains to describe the proof of Theorem 3.10. This essentially amounts to computing various derivatives of

$$(3.15) \quad Z(x, y, t) = \frac{2\langle X(y, t) - X(x, t), \nu(x, t) \rangle}{\|X(y, t) - X(x, t)\|^2}.$$

To facilitate the computation, we write $d(x, y, t) = \|X(y, t) - X(x, t)\|$, $\omega(x, y, t) = X(y, t) - X(x, t)$, $\partial_{x^i} = \frac{\partial X(x, t)}{\partial x^i}$ and $\partial_{y^j} = \frac{\partial X(y, t)}{\partial y^j}$, and always work in normal coordinates at x and y , in particular we have

$$(3.16) \quad \frac{\partial}{\partial x^i} \partial_{x^j} = h_{ij}(x) \nu(x), \quad \frac{\partial}{\partial x^i} \nu(x) = -h_{ip}(x) \partial_{x^p}.$$

Lemma 3.17. *The first derivative of Z with respect to x^i is given by*

$$(3.18) \quad \frac{\partial Z}{\partial x^i} = \frac{2}{d^2} (Z \langle \omega, \partial_{x^i} \rangle - h_{ip}(x) \langle \omega, \partial_{x^p} \rangle).$$

Proof. Observe that $\frac{\partial}{\partial x^i} d^2 = -2\langle \omega, \partial_{x^i} \rangle$. Using this, equation (3.16), and the fact that $\langle \partial_{x^i}, \nu(x) \rangle = 0$, we compute

$$\begin{aligned} \frac{\partial Z}{\partial x^i} &= \frac{2}{d^2} \langle \omega, \frac{\partial}{\partial x^i} \nu(x) \rangle - \frac{2}{d^4} \langle \omega, \nu(x) \rangle \frac{\partial}{\partial x^i} d^2 \\ &= -\frac{2}{d^2} h_{ip}(x) \langle \omega, \partial_{x^p} \rangle + \frac{2}{d^2} Z \langle \omega, \partial_{x^i} \rangle. \end{aligned}$$

This proves the lemma. \square

Similarly, the first derivative of Z with respect to y^i is given by

$$(3.19) \quad \frac{\partial Z}{\partial y^i} = \frac{2}{d^2} (\partial_{y^i} \nu(x) - Z \omega).$$

Exercise 3.20 (Time derivative). *Show that*

$$(3.21) \quad \partial_t Z = -\frac{2}{d^2} (H(x) + H(y) + \langle \omega, \nabla H(x) \rangle) + Z^2 H(x).$$

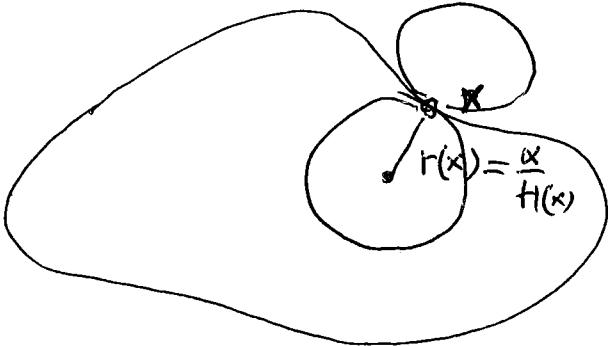
We also need the formulas for the second spatial derivatives.

Lemma 3.22. *At a critical point of Z with respect to y we have*

$$(3.23) \quad \frac{\partial^2 Z}{\partial x^i \partial y^j} = \frac{2}{d^2} (Z \delta_{ip} - h_{ip}(x)) \langle \partial_{y^j}, \partial_{x^p} \rangle - \frac{2}{d^2} \frac{\partial Z}{\partial x^i} \langle \partial_{y^j}, \omega \rangle.$$

(1)

recall:



α -noncollapsing
(quantitative
embeddedness)

$$Z(x, y, t) = \frac{2 \langle F(y, t) - F(x, t), r(x, t) \rangle}{\|F(y, t) - F(x, t)\|^2}$$

$$Z_*(x, t) = \inf_{y \neq x} Z(x, y, t)$$

$$\text{Thm* } \partial_t Z_* \geq \Delta Z_* + |A|^2 Z_*$$

For the proof we have to compute various derivatives:

~~Proof.~~ •) $\frac{\partial Z}{\partial x^i} = \frac{2}{\|F(y, t) - F(x, t)\|^2} \left(Z \langle F(y, t) - F(x, t), \frac{\partial F(x, t)}{\partial x^i} \rangle - A_{ip}(x, t) \langle F(y, t) - F(x, t), \frac{\partial F(x, t)}{\partial x^p} \rangle \right)$

~~•)~~ $\frac{\partial Z}{\partial y^i} = \frac{2}{\|F(y, t) - F(x, t)\|^2} \left\langle \frac{\partial F(y, t)}{\partial y^i}, r(x, t) - Z \cdot (F(y, t) - F(x, t)) \right\rangle$

~~•)~~
$$\begin{aligned} \frac{\partial^2 Z}{\partial t^2} &= - \frac{2}{\|F(y, t) - F(x, t)\|^2} \left(H(x, t) + \right. \\ &\quad \left. + \langle F(y, t) - F(x, t), \nabla H(x, t) \rangle \right) \\ &\quad + Z^2 H(x) \end{aligned}$$

•) At a critical point of Z wrt y :

(2)

$$\begin{aligned} \frac{\partial^2 Z}{\partial x^i \partial y^j} &= \frac{2}{\|F(y_i, t) - F(x, t)\|^2} \left\langle \frac{\partial F(y_i, t)}{\partial y^j}, \frac{\partial}{\partial x^i} (r(x, t) - Z(x, y_i, t)(F(y_i, t) - F(x, t))) \right\rangle \\ &= - \frac{2}{\|F(y_i, t) - F(x, t)\|^2} \left(\left\langle \frac{\partial F(y_i, t)}{\partial y^j}, A_{ip}(x, t) \frac{\partial F(x, t)}{\partial x^p} \right\rangle \right. \\ &\quad \left. + \frac{\partial Z}{\partial x^i} \left\langle \frac{\partial F(y_i, t)}{\partial y^j}, F(y_i, t) - F(x, t) \right\rangle \right) \\ &\Rightarrow -Z \left\langle \frac{\partial F(y_i, t)}{\partial y^j}, \frac{\partial F(x, t)}{\partial x^i} \right\rangle \end{aligned}$$

•) Similarly (always at a crit. point ∇Z in ON basis):

$$\frac{\partial^2 Z}{\partial y^i \partial y^j} = - \frac{2}{\|F(y_i, t) - F(x, t)\|^2} (Z \delta_{ij} + A_{ij}(y_i, t))$$

Indeed, note that at a crit. point:

$$r(x, t) - Z \cdot (F(y_i, t) - F(x, t)) \text{ is parallel to } r(y_i, t)$$

$$\text{and } \|r(x, t) - Z \cdot (F(y_i, t) - F(x, t))\|^2$$

$$= 1 - \underbrace{2Z \langle r(x, t), F(y_i, t) - F(x, t) \rangle}_{= 2\|F(y_i, t) - F(x, t)\|^2} + Z^2 \|F(y_i, t) - F(x, t)\|^2$$

$$\text{thus } \underbrace{r(x, t) - Z \cdot (F(y_i, t) - F(x, t))}_{=} = -r(y_i, t) \quad (*)$$

$$\begin{aligned}
 \bullet) \quad & \frac{\partial^2 Z}{\partial x^i \partial x^j} = \frac{2}{\|F(y,t) - F(x,t)\|^2} \left\langle F(y,t) - F(x,t), \frac{\partial F}{\partial x^j}(x,t) \right\rangle \frac{\partial Z}{\partial x^i} \quad (3) \\
 & + \frac{2}{\|F(y,t) - F(x,t)\|^2} \frac{\partial Z}{\partial x^j} \left\langle F(y,t) - F(x,t), \frac{\partial F}{\partial x^i}(x,t) \right\rangle \\
 & - \frac{2}{\|F(y,t) - F(x,t)\|^2} Z \delta_{ij} \\
 & + Z^2 A_{ij}(x,t) \\
 & - \frac{2}{\|F(y,t) - F(x,t)\|^2} \underset{(Codazzi)}{\nabla_p} A_{ij}(x,t) \left\langle F(y,t) - F(x,t), \frac{\partial F}{\partial x^p}(x,t) \right\rangle \\
 & + \frac{2}{\|F(y,t) - F(x,t)\|^2} A_{ij}(x) \\
 & - Z A_{ip}(x) A_{pj}(x)
 \end{aligned}$$

$\bullet)$ ~~Exer~~ At a critical point:

$$\begin{aligned}
 \frac{\partial Z}{\partial t} = & - \frac{2}{\|F(y,t) - F(x,t)\|^2} \left(H(x,t) + H(y,t) \right. \\
 & \left. + \left\langle F(y,t) - F(x,t), \nabla H(x,t) \right\rangle \right) \\
 & + Z^2 H(x)
 \end{aligned}$$

(4)

Proof of Thm. Want to show Z_* is a viscosity supersolution $\partial_t Z_* \geq \Delta Z_* + |A|^2 Z_*$

Given any (x, t) and any $\phi = \phi(x, t) \in C^2$

with $\phi \leq Z_*$ in backwards par. nbd of (x, t) and $\phi = Z_*$ at (x, t) , we have to show

$$\partial_t \phi \geq \Delta \phi + |A|^2 \phi.$$

Let y be a point where \inf in the def of Z_* is attained. Then:

$$0 \leq -\partial_t(Z - \phi) + \sum_{i=1}^n \partial_{x_i} \partial_{x_i}(Z - \phi) + 2\partial_{x_i} \partial_{y_i}(Z - \phi) + \partial_{y_i} \partial_{y_i}(Z - \phi)$$

~~$\Rightarrow \partial_t \phi = \Delta \phi - |A|^2 \phi + |A|^2 Z_*$~~

~~$+ \frac{2}{d^2} (H(x) + H(y) + \langle F(y) - F(x), \nabla H(x) \rangle) - Z^2 H(x)$~~

$$0 \leq \partial_t \phi - \Delta \phi - |A|^2 \phi + \cancel{|A|^2 \epsilon}$$

$$+ \frac{2}{d^2} \left(H(x) + \tilde{H}(y) + \langle F(y) - F(x), PH(x) \rangle \right) - \cancel{\sum_{ij} A_{ij} \epsilon(x)}$$

$$+ \frac{2}{d^2} \left(\langle F(y) - F(x), \partial_{xj} F \rangle \partial_{xi} \epsilon + \cancel{\langle F(y) - F(x), \partial_{xj} F \rangle \partial_{xi} \epsilon} \right)$$

$$+ \cancel{\sum_{ij} A_{ij}}(x) + \frac{2}{d^2} (A_{ij}(x) - R_{ij}) = \cancel{\sum_i A_{ip}(x) A_{pi}(x)}$$

$$- \cancel{\frac{2}{d^2} \langle F(y) - F(x), \partial_{xp} F \rangle \nabla_p A_{ij}(x)}$$

$$- \frac{4}{d^2} A_{ip}(x) \langle \partial_{yi} F, \partial_{xp} F \rangle$$

$$- \frac{4}{d^2} \langle \partial_{yi} F, F(y) - F(x) \rangle \partial_{xi} \epsilon$$

$$+ \frac{4}{d^2} \cancel{\sum_i} \langle \partial_{yi} F, \partial_{xi} F \rangle$$

$$\star - \frac{2}{d^2} \left(\cancel{\sum_i} \epsilon_{ij} + \tilde{A}_{ij}(y) \right)$$

(6)

$$0 \leq \partial_t \phi - \Delta \phi - |A|^2 \phi$$

$$\begin{aligned}
& + \frac{4}{d^2} H(x) - \frac{4}{d^2} A_{ip}(x) \langle \partial_{y^i} F, \partial_{x^p} F \rangle \\
& - \frac{4n}{d^2} Z + \frac{4}{d^2} Z \langle \partial_{y^i} F, \partial_{x^i} F \rangle \\
& + \frac{4}{d^2} \langle F(y) - F(x), \partial_{x^i} F - \partial_{y^i} F \rangle \partial_{x^i} Z
\end{aligned}$$

$$0 \leq \partial_t \phi - \Delta \phi - |A|^2 \phi$$

$$\begin{aligned}
& + \frac{4}{d^2} \left(A_{ip}(x) - Z \delta_{ip} \right) \left(\delta_{ip} - \langle \partial_{y^i} F, \partial_{x^p} F \rangle + \frac{2}{d^2} \langle F(y) - F(x), \partial_{x^p} F \rangle \right) \\
& \underbrace{\geq 0 \text{ by}}_{\text{def of } Z_*} \quad \underbrace{\geq 0 \text{ (Excr)}}_{\langle F(y) - F(x), \partial_{y^i} F - \partial_{x^i} F \rangle}
\end{aligned}$$

$$\Rightarrow 0 \leq \partial_t \phi - \Delta \phi - |A|^2 \phi$$

□