

(1)

Thm Suppose $M^n \subset \mathbb{R}^{n+1}$ is a ^{convex} complete embedded hypersurface such that $\frac{\lambda_1}{H} \geq \varepsilon$, $|VH| \leq CH^2$

Then M is compact. $\text{diam } M \leq \frac{1}{C} (e^{\pi C/\varepsilon} - 1) \cdot (\frac{\text{sup } H}{H})^{-1}$

Rmk •) Hamilton has a theorem without assuming (*).

-) (*) holds for any blowup limit of a mean convex MCF (see next week)

Proof Fix $p \in M$, wlog $H(p) = 1$. Let $D < \infty$.

Set $M_D = \{q \in M : d(q, p) \leq D\}$

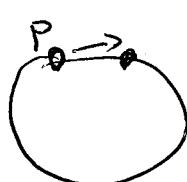
Claim If $D \geq \frac{1}{C} (e^{\pi C/\varepsilon} - 1)$, then $M_D = M$.

We will show that the

Gauss map $r : M_D \rightarrow S^n$ is surjective.
 $q \mapsto r(q)$

Let $w \in S^n \setminus \{\pm r(p)\}$.

Consider the ODE



$$\begin{cases} \dot{y} = \frac{\omega^\top}{\|\omega^\top\|} \\ y(0) = p \end{cases} \quad \text{where } \omega^\top(q) = \omega - \langle \omega, r(q) \rangle r(q)$$

Note that

$$\begin{aligned} \frac{d}{ds} \langle r, \omega \rangle &= \sum_{i=1}^n \langle \dot{r}, e_i \rangle \langle \nabla e_i r, \omega \rangle = \frac{1}{\|\omega^\top\|} \sum_{ij=1}^n A_{ij} \langle \omega, e_i \rangle \\ &\geq \frac{1}{\|\omega^\top\|} \varepsilon H \|\omega^\top\|^2 = \varepsilon H \sqrt{1 - \langle r, \omega \rangle} \end{aligned}$$

$$\Rightarrow \frac{d}{ds} \arcsin(r, \omega) \geq \varepsilon H. \quad (2)$$

Suppose $\gamma(s)$ exists for $s \in [0, D]$. Then:

$$\begin{aligned} \overline{\Pi} &> \arcsin(r(\gamma(D)), \omega) - \arcsin(r(p), \omega) \\ &\geq \varepsilon \int_0^D H(\gamma(s)) ds \geq \varepsilon \int_0^D \frac{1}{1+Cs} ds \\ &= \underline{\frac{\varepsilon}{C} \ln(1+CD)} \end{aligned}$$

$\frac{d}{ds} H = \nabla H \cdot \frac{d\gamma}{ds} \stackrel{(*)}{\geq} -CH^2$
 $\Rightarrow H(\gamma(s)) \geq \frac{1}{1+Cs}$

↳ if $D \geq \frac{1}{C}(e^{\Pi C/\varepsilon} - 1)$

$$\Rightarrow \exists s^* \in [0, D] \text{ st } \gamma(s) \rightarrow q^* \in M_D, \quad r(q^*) = \omega$$

The assertion follows.



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Type I and type II singularities

$\{M_t\}_{t \in [0, T]}$ MCF of closed hypersurfaces on maximal time interval.

$$\text{Prop} \quad \max_{p \in M} |A(p, t)| \geq \frac{1}{\sqrt{2(T-t)}}$$

$$\text{Proof} \quad \partial_t |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 \leq \Delta |A|^2 + 2|A|^4$$

Compare with the ODE for $\varphi(t) = \max_{p \in M} |A(p, t)|^2$

$$\frac{d}{dt} \varphi \leq 2\varphi^2$$

$$\Rightarrow -\frac{d}{dt} \frac{1}{\varphi} \leq 2 \Rightarrow \frac{1}{\varphi(t)} - \frac{1}{\varphi(s)} \leq 2(s-t) \quad (0 < t < s < T)$$

$$\text{Max.princ.} \Rightarrow \frac{1}{\max|A(\cdot, t)|^2} - \frac{1}{\max|A(\cdot, s)|^2} \leq 2(s-t)$$

Apply for $s_i \nearrow T$ with $\max|A(\cdot, s_i)| \rightarrow \infty$

$$\Rightarrow \frac{1}{\max|A(\cdot, t)|^2} \leq 2(T-t) \quad \blacksquare$$

(4)

Def Let $\{M_t\}_{t \in [0, T]}$ be a MCF of closed hypersurfaces on a maximal time interval.

We say that the flow develops a type I singularity at time T , if $\exists C < \infty$:

$$\max_{M_t} |A| \leq \frac{C}{\sqrt{T-t}}$$

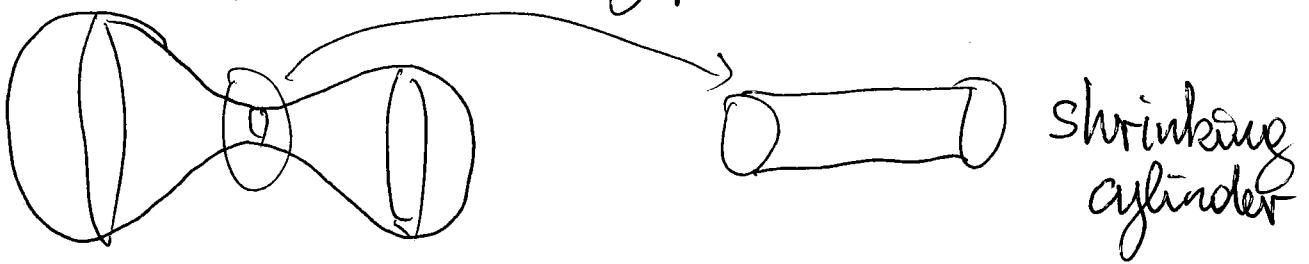
and a type II singularity if

$$\limsup_{t \rightarrow T} \max_{M_t} |A| \sqrt{T-t} = \infty$$

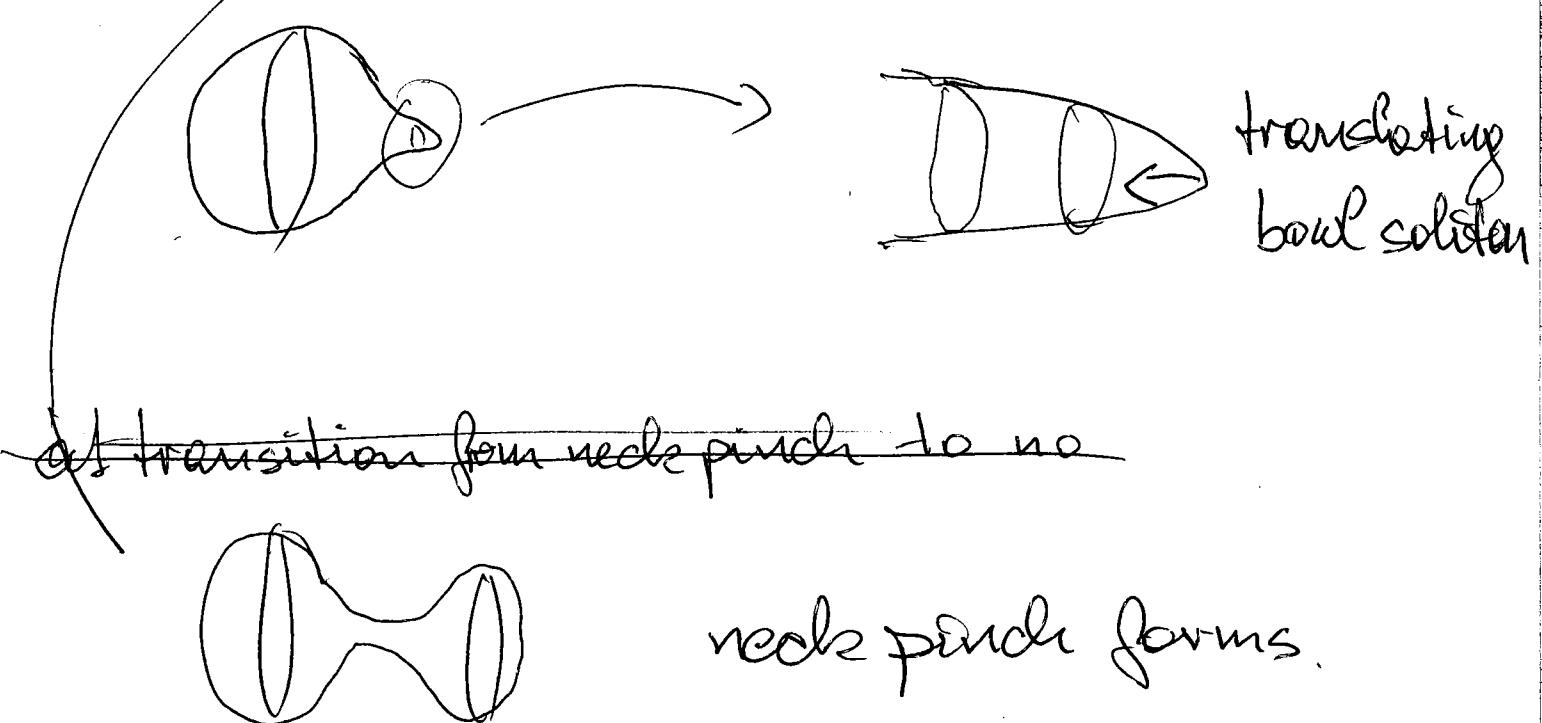
Rank type II \Leftrightarrow not type I.

Ex neck pinch is type I

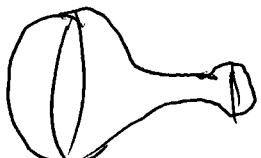
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Ex degenerate neckpinch is type II



somewhere
in between



degenerate neckpinch forms.

no singularity forms.

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Note: In the mean convex case $|A|$ & H blow up at the same rate.

Indeed, $H^2 \leq n|A|^2$ always true by Cauchy-Schwarz.

On the other hand $|A|^2 \leq CH^2$ at $t = 0$.

Exer Use the maximum principle, to show

that $|A|^2 \leq CH^2 \quad \forall t \in [0, T]$.

Type II Blowup Limits

①

$\{M_t\}_{t \in [0, T]}$ NCF of closed hypersurfaces

Assume $\limsup_{t \nearrow T} \max_{p \in \hat{M}} |A(p, t)| \sqrt{T-t} = \infty$ (type II)

Select $t_k \in [0, T - \frac{1}{k}]$, $p_k \in \hat{M}$ such that

$$|A(p_k, t_k)|^2 (T - \frac{1}{k} - t_k) = \max_{\substack{t \in [T - \frac{1}{k}] \\ p \in \hat{M}}} |A(p, t)|^2 (T - \frac{1}{k} - t)$$

(type II) $\Rightarrow C_k \rightarrow \infty$, $t_k \rightarrow T$.

Thus can find $t_k \nearrow T$ such that

$$\lambda_k := |A|(p_k, t_k) \nearrow \infty$$

$$|A|^2(p_k, t_k)(T - \gamma_k - t_k) \rightarrow \infty$$

Moreover, can assume $p_k \rightarrow \hat{p} \in \hat{M}$.

Consider the rescaled flows

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$$F^k : \tilde{M} \times I_k \rightarrow \mathbb{R}^{n+1}$$

$$\text{where } F^k(p, t) = \lambda_k \cdot \left(F\left(p, \frac{t}{\lambda_k^2} + t_k \right) - F(p_k, t_k) \right)$$

$$\text{and } I_k = [a_k, b_k]$$

$$\text{Note that: } \bullet) \frac{a_k}{\lambda_k^2} + t_k = 0 \Rightarrow a_k = -\lambda_k^2 t_k \rightarrow -\infty$$

$$\bullet) \frac{b_k}{\lambda_k^2} + t_k = T \Rightarrow b_k = \lambda_k^2(T - t_k) \rightarrow +\infty$$

$$\Rightarrow \underline{A^k(p_k, 0)}$$

$$\bullet) F^k(p_k, 0) = 0 \in \mathbb{R}^{n+1} \quad \cancel{\text{and } A^k(p_k, 0) = 1}$$

$$\text{and } |A^k|(p_k, 0) = 1.$$

(3)

ClaimFor every $\varepsilon > 0$, $\Lambda < \infty \exists \bar{k} \in \mathbb{N}$:

$$\max_{p \in M} |A^k|(p, t) \leq 1 + \varepsilon$$

for every $k \geq \bar{k}$ and $t \in [-\lambda_k^2 t_0, \bar{\lambda}]$

~~Proof $|A_k|^2(p, t) = \frac{1}{|A|^2(p_k, t_k)} |A|(p, t)$~~

~~Proof~~ $|A_k|^2(p, t) = \frac{1}{\lambda_k^2} |A|^2(p, \frac{t}{\lambda_k^2} + t_k)$
 $\leq \frac{1}{\lambda_k^2} \frac{\overbrace{|A|^2(p_k, t_k)}^{=\lambda_k^2} (T - \frac{1}{\lambda_k} - t_k)}{(T - \frac{1}{\lambda_k} - t_k - t/\lambda_k^2)}$ (by selection of (p_k, t_k))
 $= \frac{\lambda_k^2 (T - \frac{1}{\lambda_k} - t_k)}{\underbrace{\lambda_k^2 (T - \frac{1}{\lambda_k} - t_k)}_{\rightarrow \infty} - t} \Rightarrow \text{Claim } \square$

(4)

Thus along a subsequence $k_i \rightarrow \infty$ we can

pass to a limit $F^\infty: \hat{\Omega}^\infty \times (-\infty, \infty) \rightarrow \mathbb{R}^{n+1}$

The limit flow is:

-) eternal (i.e. defined for $t \in (-\infty, \infty)$)
-) nonempty
-) satisfies $|A| \leq 1$ and $|A|(0, 0) = 1$.

Rmk: In mean convex case, can show it is a translating solution using convexity estimate & Hamilton's Harnacking.

Thm (Huisken) If $M_0^n \subset \mathbb{R}^{n+1}$ is a convex closed embedded hypersurface, then the MCF starting at M_0 converges to a round point.

Proof (modern version)

Let $\{M_t\}_{t \in [0, T]}$ be the MCF of M_0 defined on a max. time interval.

We know $\frac{\lambda_1}{H} \geq \varepsilon$.

Flow becomes singular at time T .

~~assume~~ If (Type II) \Rightarrow get eternal flow as type I blowup limit

but $\frac{\lambda_1}{H} \geq \varepsilon$ & gradient est \Rightarrow compactly eternal

\Rightarrow must be (type I)

(5)

Let $x_0 \in \mathbb{R}^{n+1}$ be a blowup point.

Last week $\Rightarrow \forall \lambda_k \rightarrow \infty \exists$ subsequence λ_{k_i} st

$$M_{t'}^{\lambda_{k_i}} = \lambda_{k_i} \cdot (M_{T + \lambda_{k_i}^{-2} t'} - x_0)$$

Converges to a mean convex shrinker M_t^∞

$\frac{\lambda_1}{H} \geq \varepsilon$ (& grad est) $\Rightarrow M^\infty$ compact

~~closed~~

$$\Rightarrow M_{t'}^\infty = S_{\sqrt{2n|t'|}}$$

Classification
of shrinkers

Limit unique $\Rightarrow x_0$ unique & subseq. convergence
can be upgraded to full convergence

