

Preserved curvature conditions

(1)

recall: $\partial_t H = \Delta H + |A|^2 H \Rightarrow H \geq 0$ preserved along MCF.
max. principle

now consider: $\partial_t A_{ij} = \Delta A_{ij} + |A|^2 A_{ij} - 2H A_{ik} g^{kl} A_{lj}$

Goal: Want to show $A_{ij} \geq 0$ preserved along MCF
i.e. $A_{ij} v^i v^j \geq 0 \quad \forall v.$

Tensor maximum principle

$$\partial_t M_{ij} = \Delta M_{ij} + u^k \nabla_k M_{ij} + N_{ij} \text{ on closed mfld.}$$

where $N = N(M, g, t)$ locally Lipschitz.

Def: We say that N satisfies the
null eigenvector condition

$$\text{if } M_{ij} v^j = 0 \Rightarrow N_{ij} v^i v^j \geq 0.$$

(2)

Thm If N satisfies the null eigenvector condition,
 then $M_{ij} \geq 0$ at $t=0 \Rightarrow M_{ij} \geq 0 \quad \forall t \geq 0$.

Proof will show $M_{ij} \geq 0$ on $0 \leq t \leq \delta$.

$$\tilde{M}_{ij} := M_{ij} + \varepsilon(\delta+t) g_{ij}$$

$$\tilde{M}_{ij} > 0 \text{ on } 0 \leq t \leq \delta \quad (\forall \varepsilon > 0)$$

Claim:

If not, \exists first time $\theta \in (0, \delta]$ st \tilde{M}_{ij} acquires
 a. null eigenvector v ($|v|=1$) at some point x .
 b. null eigen. cond $\Rightarrow N_{ij} v^i v^j \geq 0$

$$|\tilde{N} - N| \leq C |\tilde{M} - M| \leq C \varepsilon \delta$$

$$\Rightarrow \underbrace{N_{ij} v^i v^j}_{\text{Lip.}} \geq -C \varepsilon \delta$$

Extend v to a vector field \tilde{v}

$$\nabla v = 0 \quad \text{at } x.$$

$$\partial_t v = 0.$$

$$\text{Consider } f := \tilde{M}_{ij} v^i v^j$$

Then $f \geq 0$ on $[0, \theta]$, and $f(x, \theta) = 0$

and $\partial_t^2 f \leq 0, \quad \nabla f = 0, \quad \Delta f \geq 0 \quad \text{at } (x, \theta)$.

$$\partial_t f = (\partial_t M_{ij}) v^i v^j + \varepsilon - 2\varepsilon(\delta+t) + A_{ij} v^i v^j \quad (3)$$

$$\geq \Delta M_{ij} v^i v^j + u^k \nabla_k M_{ij} v^i v^j + \underbrace{\varepsilon - 4\varepsilon \delta C}_{\geq \varepsilon/2}$$

$$+ N_{ij} v^i v^j$$

At (x, θ) :

provided $\delta \leq \frac{1}{8C}$

$$\begin{aligned} \cdot) \quad \nabla_k f &= \nabla_k \tilde{M}_{ij} v^i v^j + \tilde{M}_{ij} \underbrace{\nabla_k v^i}_{=0} v^j + \tilde{M}_{ij} v^i \underbrace{\nabla_k v^j}_{=0} \\ &= \nabla_k M_{ij} v^i v^j \end{aligned}$$

$$\cdot) \quad \Delta f = \Delta M_{ij} v^i v^j$$

hence

$$0 \geq \partial_t f \geq \underbrace{\Delta f}_{\geq 0} + \underbrace{\langle u, \nabla f \rangle}_{=0} + \varepsilon/2 + N_{ij} v^i v^j$$

$$\Rightarrow \underbrace{N_{ij} v^i v^j}_{\leq -\varepsilon/2} \quad \square$$



Thm $A_{ij} \geq 0$ is preserved along MCF (4)

Proof $N_{ij}(A, g, t) := |A|^2 A_{ij} - 2H A_{ik} g^{kl} A_{lj}$

If $A_{ij} v^j = 0$, then $N_{ij} v^i v^j \geq 0$

tensor max. princ. \Rightarrow assertion \square

Note: If M_0 satisfies $A_{ij} \geq 0$, then $\exists \beta > 0$

st $A_{ij} \geq \beta H g_{ij}$

(i.e. $A_{ij} - \beta H g_{ij} \geq 0$)

Thm $A_{ij} \geq \beta H g_{ij}$ is preserved along MCF. (5)

Proof

$$\begin{aligned} & \partial_t \left(\underbrace{A_{ij} - \beta H g_{ij}}_{=: M_{ij}} \right) \\ &= \Delta \left(\underbrace{A_{ij} - \beta H g_{ij}}_{=: M_{ij}} \right) \\ &+ |A|^2 (A_{ij} - \beta H g_{ij}) - 2H A_{ik} g^{kl} A_{ej} + 2\beta H^2 A_{ij} \end{aligned}$$
$$=: N_{ij}$$

If $M_{ij} v^i = 0$, i.e. $Av = \beta Hv$, then

$$N_{ij} v^i v^j = 0 - 2\beta^2 H^3 |v|^2 + 2\beta^2 H^3 |v|^2 = 0,$$

tensor max. principle $\Rightarrow M_{ij} \geq 0 \quad \forall t \geq 0$



Rank Denote by $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ the principal curvatures (i.e. eigenvalues of A) ordered by size. (6)

$$(k=1) \quad A_{ij} \geq 0 \Leftrightarrow \lambda_1 \geq 0, \dots, \lambda_n \geq 0$$

$$(k=n) \quad H \geq 0 \Leftrightarrow \lambda_1 + \dots + \lambda_n \geq 0$$

Def A hypersurface is k -convex if

$$\lambda_1 + \dots + \lambda_k \geq 0$$

and β -uniformly k -convex if

$$\lambda_1 + \dots + \lambda_k \geq \beta H$$

Also preserved under MCF

(proof uses that

$$(\lambda_1 + \dots + \lambda_k)(p) = \min_{\substack{e_1, \dots, e_k \in T_p M \\ \langle e_i, e_j \rangle = \delta_{ij}}} \{ A_p(e_1, e_1) + \dots + A_p(e_k, e_k) \}$$

is concave.

ODE leaves
convex cone
invariant

$\sim \lambda_1 + \dots + \lambda_k \geq \beta H$ describes convex cone
in space of symmetric matrices, $\partial_t A^i_j = \Delta A^i_j + (A^T A)^i_j$