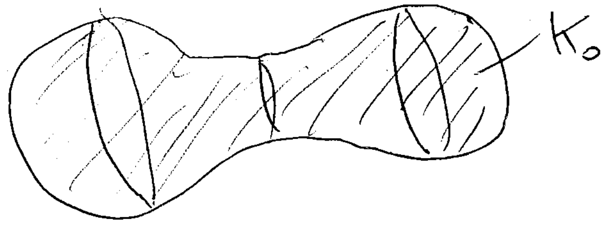
recall: $M_0 = \partial K_0 \subset \mathbb{R}^{n+1}$ closed mean convex hypersurface

$\Rightarrow \exists!$ MCF $\{K_t\}_{t\geq 0}$ starting at K_0.



•) $K_{t_2} \subset K_{t_1}$ for $t_2 > t_1$ (mean convex)

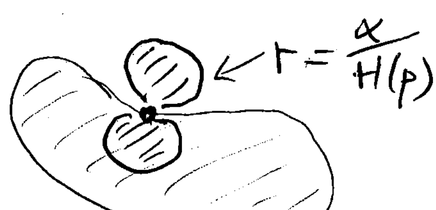
•) $K_t = \emptyset$ for $t \geq T_{ext}$

•) $\{K_t\}_{t\geq 0}$ is the maximal family of closed sets starting at K_0 that satisfies the avoidance principle

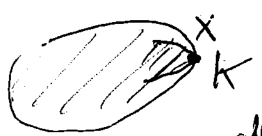
•) $\begin{cases} -\operatorname{div}\left(\frac{Du}{|Du|}\right) = \frac{1}{|Du|} & \text{in } K_0 \\ u = 0 & \text{on } \partial K_0 \end{cases}$ $\quad u(x) = t \Leftrightarrow x \in \partial K_t$

•) $\mathcal{H}^n \llcorner \partial K_t$ is a Brakke flow (with equality)

•) If K_0 is α-noncollapsed, then so is K_t for all $t \geq 0$.



•) $\{K_t \times \mathbb{R}\}_{t\geq 0}$ is a limit of smooth flows.



viscosity mean curvature

$H_K(p) := \inf\{ H_{\partial X}(p) \mid X \subseteq K \text{ smooth}, p \in \partial X\}$

~~$= \{ +\infty$ p not s~~ $(\inf \emptyset := +\infty)$

Proof $\operatorname{div}\left(\frac{Du^\varepsilon}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}\right) = -\frac{1}{\sqrt{\varepsilon^2+|Du^\varepsilon|^2}}$

i.e. $N^\varepsilon = \operatorname{graph}\left(\frac{u^\varepsilon}{\varepsilon}\right)$ satisfies $\vec{H} = -\frac{1}{\varepsilon}e_{n+2}$
$\subset \mathbb{R}^{n+1}\times\mathbb{R}$

$Z^*, Z_*: N^\varepsilon \to \mathbb{R}$, $Z^*(x) = \sup_{y\neq x} Z(x,y)$, $Z_*(x) = \inf_{y\neq x} Z(x,y)$

where $Z(x,y) = \frac{2\langle x-y, \nu_x\rangle}{|x-y|^2}$.

$\Delta\frac{Z^*}{H} + 2\langle\nabla \log H, \nabla\frac{Z^*}{H}\rangle \geq 0 \Rightarrow \max\frac{Z^*}{H}$ is attained at ∂N^ε

similarly, $\min\frac{Z_*}{H}$ is attained at ∂N^ε.

$\Rightarrow N^\varepsilon$ is α_ε-noncollapsed, with $\liminf_{\varepsilon\to 0}\alpha_\varepsilon \geq \alpha = \alpha(K_0)$

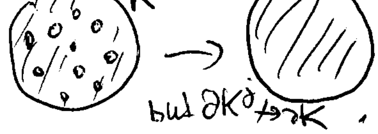
$\varepsilon\to 0 \Rightarrow \{K_t\times\mathbb{R}\}$ and thus $\{K_t\}$ is α-noncollapsed $\boxtimes$

Def The class of α-noncollapsed flows is the smallest class of set flows $\{K_t\}$ which ~~contains:~~

- ~~contains~~ all compact level set flows with smooth α-noncollapsed initial condition
- ~~contains~~ all smooth α-noncoll. flows in $U\subset\mathbb{R}^{n+1}$ open
- is closed under restriction, parabolic rescaling, Hausdorff limits

Lemma: $K^j \xrightarrow{\text{Hausdorff}} K$ $\Rightarrow \partial K^j \to \partial K$
α-noncollapsed

Rmk not true without α-noncoll, e.g.



Thm (Local curvature estimate)

$\forall \alpha > 0 \ \exists \rho = \rho(\alpha) > 0,\ C_\ell = C_\ell(\alpha) < \infty$:

If $\mathcal{K}$ is an α-noncollapsed flow in $P(p,t,r)$

$p \in \partial K_t,\ H(p,t) \leq r^{-1}$

then $\sup\limits_{P(p,t,\rho r)} |\nabla^\ell A| \leq C_\ell r^{-(\ell+1)}$.

Rmks ~~) $\mathcal{K}$ flow in~~

•) $H(p,t) \leq 1$ $\cancel{\Rightarrow |H| \leq C}$

$\Rightarrow H \leq C_0$ in $P(p,t,\rho)$ ~~parabolic ball of definite size~~

•) like "local Harnack inequality"

•) Cor: $|\nabla H| \leq C(\alpha) \cdot H^2$ (used some time ago)

i.e. curv. control at a single point, gives curv. control on par. ball of definite size.

4

$H(0,0) \leq j^{-1}$, but such that

$$\sup_{P(0,0,1)} |A| \geq j. \tag{4.5}$$

We can choose coordinates such that the outward normal of K_0^j at $(0,0)$ is e_{n+1}. Furthermore, by [HK13a, App. D] we can assume that the sequence is admissible, i.e. that for every $R < \infty$ some time slice $K_{t_j}^j$ contains $B(0,R)$, for j sufficiently large.

Claim 4.6. *The sequence of mean curvature flows $\{\mathcal{K}^j\}$ converges in the pointed Hausdorff topology to a static halfspace in $\mathbb{R}^{n+1} \times (-\infty, 0]$, and similarly for their complements.*

Proof of Claim 4.6. For $R < \infty$, $d > 0$ let $\bar{B}_{R,d} = \overline{B((-R+d)e_{n+1}, R)}$, so $\bar{B}_{R,d}$ is the closed R-ball tangent to the horizontal hyperplane $\{x_{n+1} = d\}$ at the point $d\, e_{n+1}$. When R is large, it will take time approximately Rd for $\bar{B}_{R,d}$ to leave the upper halfspace $\{x_{n+1} > 0\}$. Since $0 \in \partial K_0^j$ for all j, it follows that $\bar{B}_{R,d}$ cannot be contained in the interior of K_t^j for any $t \in [-T, 0]$, where $T \simeq Rd$. Thus, for large j we can find $d_j \leq d$ such that $\bar{B}_{R,d_j}$ has interior contact with K_t^j at some point q_j, where $\langle q_j, e_{n+1} \rangle < d$, $\|q_j\| \lesssim \sqrt{Rd}$, and moreover $\liminf_{j\to\infty} \langle q_j, e_{n+1} \rangle \geq 0$.

The mean curvature satisfies $H(q_j, t) \leq \frac{n}{R}$. Since K_t^j satisfies the α-Andrews condition, there is a closed ball $\bar{B}_j$ with radius at least $\frac{\alpha R}{n}$ making exterior contact with K_0^j at q_j. By a simple geometric calculation, this implies that K_t^j has height $\lesssim \frac{d}{\alpha}$ in the ball $B(0, R')$ where R' is comparable to $\sqrt{Rd}$. As d and R are arbitrary, this implies that for any $T > 0$, and any compact subset $Y \subset \{x_{n+1} > 0\}$, for large j the time slice K_t^j is disjoint from Y, for all $t \geq -T$.

Finally, observe that for any $T > 0$ and any compact subset $Y \subset \{x_{n+1} < 0\}$, the time slice K_t^j contains Y for all $t \in [-T, 0]$, and large j, because K_{-T}^j contains a ball whose forward evolution under MCF contains Y at any time $t \in [-T, 0]$. This proves the claim. □

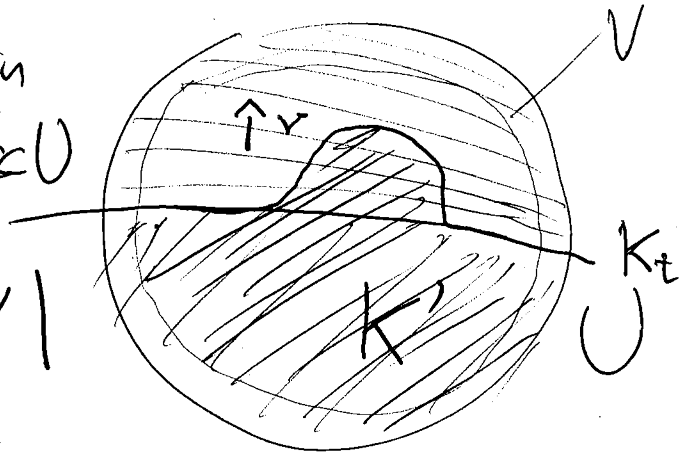
Finishing the proof of the theorem, by Claim 4.6, admissibility, and one-sided minimization (see below), we get for every $\varepsilon > 0$, every $t \leq 0$ and every ball $B(x,r)$ centered on the hyperplane $\{x_{n+1} = 0\}$, that

$$|\partial K_t^j \cap B(x,r)| \leq (1+\varepsilon)\omega_n r^n \,, \tag{4.7}$$

for j large enough. Hence, the local regularity theorem for the mean curvature flow (Theorem 2.14) implies $\limsup_{j\to\infty} \sup_{P(0,0,1)} |A| = 0$; this contradicts (4.5). □

One sided minimization

$\{K_{t'}\}_{t'\le t}$ mean convex st $\partial K_{t'}$ foliates $U\setminus \mathrm{Int}(K_t)$



Claim If $K' \supseteq K_t$ is closed domain which agrees with K_t outside $V \subset\subset U$

then $|\partial K_t \cap V| \le |\partial K' \cap V|$

Proof ~~ν outw. unit~~

ν = VF on $U\setminus \mathrm{Int}(K_t)$ defined by outward unit normal of the foliation

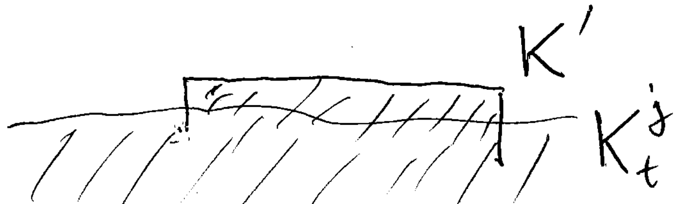
$\operatorname{div} \nu = H \ge 0$

$$\Rightarrow\ |\partial K' \cap V| - |\partial K_t \cap V| \ge \int_{\partial K' \cap V} \langle \nu, \nu_{\partial K'}\rangle - \int_{\partial K_t \cap V} \langle \nu, \nu_{\partial K_t}\rangle$$

$$\underset{\text{Stokes}}{=} \int_{(K'\setminus K_t)\cap V} \operatorname{div} \nu \ \ge\ 0 .$$

$\boxed{\checkmark}$

In our situation, can take $K' = K_t^j \cup \left(\overline{B}(x,r) \cap \{x_N \le \delta\}\right)$



Exercise 4.8 (One-sided minimization). *Use Stokes' theorem to prove the following. If $\{K_{t'} \subseteq U\}_{t' \leq t}$ is a smooth family of mean convex domains such that $\{\partial K_{t'}\}$ foliates $U \setminus \mathrm{Int}(K_t)$, then*

$$|\partial K_t \cap V| \leq |\partial K' \cap V| \tag{4.9}$$

for every closed domain $K' \supseteq K_t$ which agrees with K_t outside a compact smooth domain $V \subseteq U$. Using this, prove the density bound (4.7).

Our next estimate gives pinching of the curvatures towards positive.

Theorem 4.10 (Convexity estimate [HK13a]). *For all $\varepsilon > 0$, $\alpha > 0$, there exists $\eta = \eta(\varepsilon, \alpha) < \infty$ with the following property. If $\mathcal{K}$ is an α-Andrews flow in a parabolic ball $P(p, t, \eta r)$ centered at a boundary point $p \in \partial K_t$ with $H(p,t) \leq r^{-1}$, then*

$$\lambda_1(p,t) \geq -\varepsilon r^{-1}. \tag{4.11}$$

The convexity estimate (Theorem 4.10) says that a boundary point (p,t) in an α-Andrews flow has almost positive definite second fundamental form, assuming only that the flow has had a chance to evolve over a portion of spacetime which is large compared to $H^{-1}(p,t)$. In particular, ancient α-Andrews flows $\{K_t \subset \mathbb{R}^{n+1}\}_{t\in(-\infty,T)}$ (e.g. blowup limits) are always convex; this is crucial for the analysis of singularities.

Proof of Theorem 4.10. Fix α. The α-Andrews condition implies that the assertion holds for $\varepsilon = \frac{1}{\alpha}$. Let $\varepsilon_0 \leq \frac{1}{\alpha}$ be the infimum of the ε's for which it holds, and suppose towards a contradiction that $\varepsilon_0 > 0$.

It follows that there is a sequence $\{\mathcal{K}^j\}$ of α-Andrews flows, where for all j, $(0,0) \in \partial\mathcal{K}^j$, $H(0,0) \leq 1$ and $\mathcal{K}^j$ is defined in $P(0,0,j)$, but $\lambda_1(0,0) \to -\varepsilon_0$ as $j \to \infty$. After passing to a subsequence, $\{\mathcal{K}^j\}$ converges smoothly to a mean curvature flow $\mathcal{K}^\infty$ in the parabolic ball $P(0,0,\rho)$, where $\rho = \rho(\alpha)$ is the quantity from Theorem 4.2. Note that for $\mathcal{K}^\infty$ we have $\lambda_1(0,0) = -\varepsilon_0$ and thus $H(0,0) = 1$.

By continuity $H > \frac{1}{2}$ in $P(0,0,r)$ for some $r \in (0,\rho)$. Furthermore we have $\frac{\lambda_1}{H} \geq -\varepsilon_0$ everywhere in $P(0,0,r)$. This is because every $(p,t) \in \partial\mathcal{K}^\infty \cap P(0,0,r)$ is a limit of a sequence $\{(p_j,t_j) \in \partial\mathcal{K}^j\}$ of boundary points, and for every $\varepsilon > \varepsilon_0$, if $\eta = \eta(\varepsilon,\alpha)$, then for large j, $\mathcal{K}^j$ is defined in $P(p_j, t_j, \eta H^{-1}(p_j,t_j))$, which implies that the ratio $\frac{\lambda_1}{H}(p_j,t_j)$ is bounded below by $-\varepsilon$. Thus, in the parabolic ball $P(0,0,r)$, the ratio $\frac{\lambda_1}{H}$ attains a negative minimum $-\varepsilon_0$ at $(0,0)$. Since $\lambda_1 < 0$ and $\lambda_n > 0$ the Gauss curvature $K = \lambda_1\lambda_n$ is strictly negative. However, by the equality case of the maximum principle for $\frac{\lambda_1}{H}$, the hypersurface

Regularity & Structure theory for mean convex MCF (1)

(White, Huisken-Sinestrari, H-Kleiner)

Local curvature estimate $\forall \alpha > 0 \; \exists \rho > 0, C_\ell < \infty$:

$\mathcal{K}$ α-noncollapsed flow in $P(p,t,r)$, $H(p,t) \leq r^{-1}$

$$\Rightarrow \sup_{P(p,t,\rho r)} |\nabla^\ell A| \leq C_\ell r^{-1-\ell}$$

Convexity estimate $\forall \alpha, \varepsilon > 0 \; \exists \eta < \infty$:

$\mathcal{K}$ α-noncollapsed flow in $P(p,t,\eta r)$, $H(p,t) \leq r^{-1}$

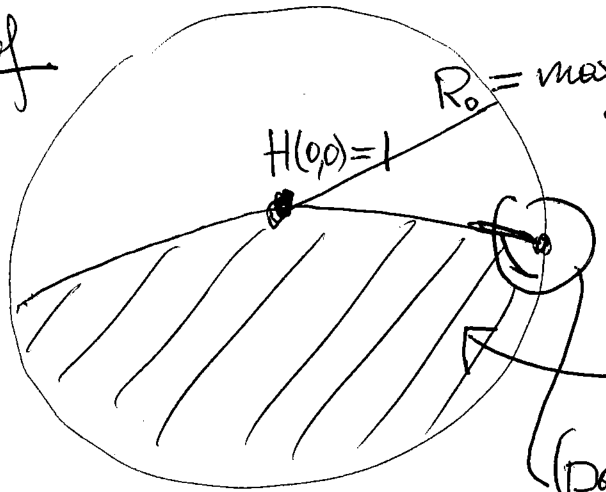
$$\Rightarrow \frac{\lambda_1}{H}(p,t) \geq -\varepsilon .$$

Global curvature estimate $\forall \alpha > 0, \Lambda < \infty \; \exists \eta, C_\ell < \infty$:

$\mathcal{K}$ α-noncollapsed flow in $P(p,t,\eta r)$, $H(p,t) \leq r^{-1}$

$$\Rightarrow \sup_{P(p,t,\Lambda r)} |\nabla^\ell A| \leq C_\ell r^{-1-\ell}$$

Idea of proof



R_0 = maximal radius where curvature stays bounded

$(R_0 \geq \rho > 0)$

smooth, convex

(part of) nonflat convex cone ↯

Singularities / high curvature regions

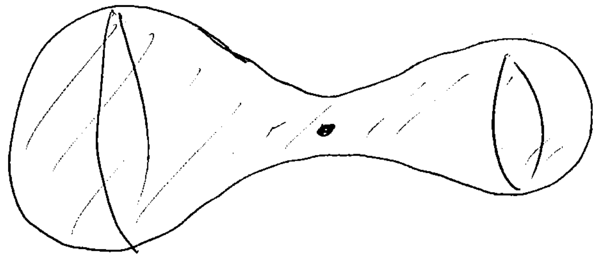
$\mathcal{M}$ α-noncollapsed flow

Blowup sequence $\mathcal{M}^j$ obtained by parabolic rescaling $(p,t) \mapsto (\lambda_j(p-p_j), \lambda_j^2(t-t_j))$ $\quad \lambda_j \to \infty$

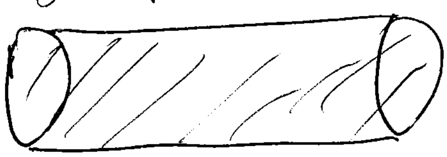
$\Rightarrow \mathcal{M}^{j'} \longrightarrow \mathcal{M}^\infty$ = limit flow

tangent flow = limit flow in special case where (p_j, t_j) is fixed.

Ex

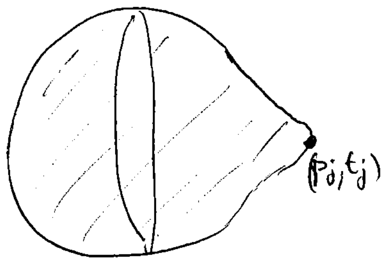


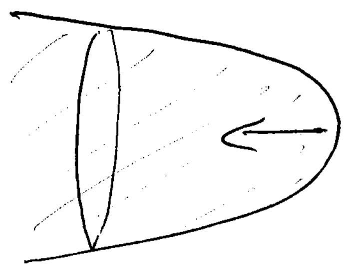
tangent flow at (p_{sing}, t_{sing}):

$\mathcal{M}^\infty =$ 

= round shrinking cylinder

Ex



$\longrightarrow \mathcal{M}^\infty =$ 

translating bowl soliton

Note: ~~I~~ ~~par~~ All limit flows are ~~in~~ particular: ~~particular~~ α-noncollapsed, ancient, defined on $\mathbb{R}^{n+1} \times (-\infty, 0)$

Structure thm for ancient α-noncollapsed flows (4)

$\mathcal{K}$ ancient α-noncollapsed flow in $\mathbb{R}^{n+1}$

~~$T \in (-\infty, +\infty]$ extinction~~

$T := \sup\{t : K_t \neq \emptyset\} \in (-\infty, +\infty]$ extinction time.

Then:

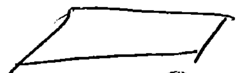
(1) $\mathcal{K} \cap \{t < T\}$ is smooth,

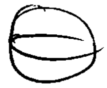
$\cancel{H(p,t) \leq}$

(2) $\mathcal{K}$ has convex time slices

(3) $\mathcal{K}$ is either a static halfspace, or it has ~~strictly positive~~ $H > 0$ and sweeps out all space, i.e. $\bigcup_{t \in \mathbb{R}} K_t = \mathbb{R}^{n+1}$

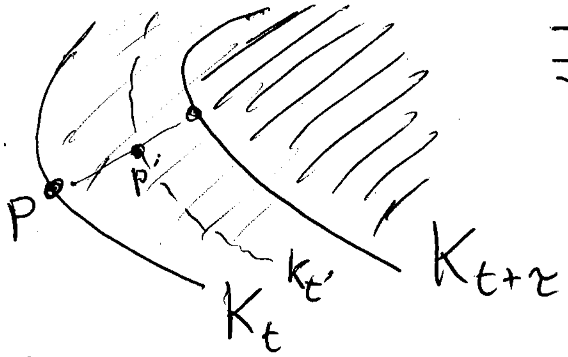
Furthermore, if $\mathcal{K}$ is backwardly selfsimilar (eg a tangent flow) then it is either

(i) a static halfspace 

or (ii) a round shrinking sphere 

or (iii) a round shrinking cylinder .

<u>Proof</u> (1) $p \in \partial K_t$, $\tau < T-t$



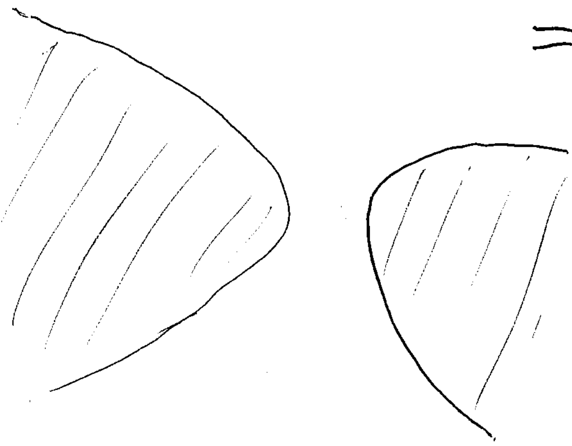
$\exists\, p' \in \partial K_{t'}$, $t' \in [t, t+\tau]$
with $|p-p'| \leq d(p, K_{t+\tau})$
and $H(p', t') \leq \frac{d(p, K_{t+\tau})}{\tau}$

glob. curv. est centered at (p', t')

$$\Rightarrow \; H(p,t) \leq C(\tau, d(p, K_{t+\tau})) .$$

(2) convexity estimate $\Rightarrow \frac{\lambda_1}{H} \geq 0$.

$\Rightarrow \partial K_t$ has positive semidefinite 2nd fundamental form



Only one connected component
$\Rightarrow$ convex .

(3) $H = 0$ at some (p,t) $\underset{\substack{\text{loc. curv. est} \\ \kappa\text{-noncoll}}}{\Rightarrow}$ static halfspace

Other case: $H > 0$. as in (2) $\Rightarrow$ sweeps out all space.

Moreover: backw. selfsimilar

$\Rightarrow t = -\frac{1}{2}$ slice satisfies $H \cancel{+} \langle x, \nu \rangle = 0$

convexity $\Rightarrow |A| \leq H \leq |x|$ grows at most linearly

one sided minimization $\Rightarrow |\partial K_t \cap B_r| \leq C r^n$

Huisken's classification $\Rightarrow$ ~~Classi~~ assertion. $\boxed{}$

Size of the singular set

$\mathcal{K}$ α-noncollapsed flow

$\mathcal{S} := \{ (p,t) \in \partial\mathcal{K} :$ ~~flow not smooth in a~~ $\nexists$ nbd of (p,t) where flow is smooth $\}$

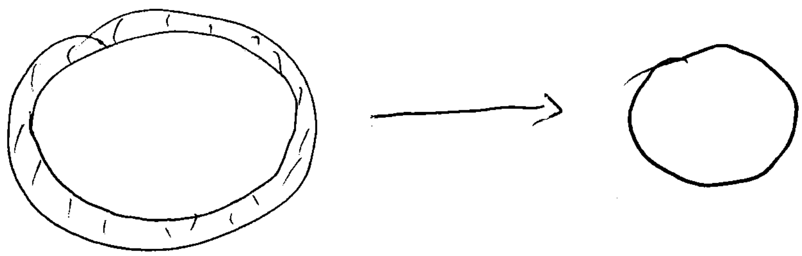
$$\subset \underbrace{\mathbb{R}^{n+1} \times \mathbb{R}}_{\text{space-time}}$$

$$d((x_1,t_1),(x_2,t_2)) := \max(|x_1 - x_2|, |t_1 - t_2|^{1/2})$$

dim = Hausdorff dim wrt d

eg $\dim(\mathbb{R}^{n+1} \times \mathbb{R}) = n+3$.

Partial regularity thm $\quad \dim \mathcal{S} \leq n-1$.

Ex  $\dim \mathcal{S} = n-1$

Proof ~~loc. curv. e~~ Str. thm & Huisken's mon. formula

$\Rightarrow$ tangent flows: static ~~halfsp~~ halfspace, shrink. sphere or cyl. (of mult. one)

loc. curv. est $\Rightarrow$ $\mathcal{S} = \{ (p,t) \in \partial\mathcal{K} :$ no tang. flow at (p,t) is static halfspace $\}$

If $\dim \mathcal{S} > n-1$ $\underset{\text{blowup at density point}}{\Rightarrow}$ $\exists$ tangent flow with $\dim \mathcal{S}^0 > n-1$