

Existence of Brakke flow via elliptic regularization (1)

More GMT:

n-current $T: C_c^\infty(\Lambda^n \mathbb{R}^N) \rightarrow \mathbb{R}$ linear, continuous.

Ex M^n surface, defines $T_M(\omega) = \int_M \omega$

T current \rightarrow mass measure μ_T , total mass $M[T]$

$$\mu_T(U) := \sup \{ T(\theta\omega) : \theta \in C_c^\infty(\Lambda^n \mathbb{O}), |\theta| \leq 1 \}$$

$$M[T] := \mu_T(\mathbb{R}^N) \quad U \subset \mathbb{R}^N \text{ open}$$

$D_n(\mathbb{R}^N)$ space of all n-currents in \mathbb{R}^N

$D_n^{\text{loc}}(\mathbb{R}^N)$ locally ~~integer n-rectifiable currents~~ ^{integral n-currents}

$$T(\omega) = \int_{\Omega} \langle \omega(x), \xi(x) \rangle \Theta(x) d\mu(x)$$

↑
\$\sum \mu\$ ↑
\$\mathbb{Z}_+\$ multiplicity
 μ integer n-rectifiable
Radon measure

$\xi(x) = e_1, \dots, e_n$ for some ONB e_1, \dots, e_n of $T_x M$
measurable choice of orientation.

Note: $\mu: D_n^{\text{loc}}(\mathbb{R}^N) \rightarrow \mathcal{I}M_n(M)$ is onto & many-to-one.

Boundary $\partial T(\omega) := T(\partial\omega)$, $\omega \in C_c^\infty(\Lambda^{n-1} \mathbb{R}^n)$ (2)

Convergence $T_i \rightarrow T \Leftrightarrow T_i(\omega) \rightarrow T(\omega)$ $\forall \omega \in C_c^\infty(\Lambda^n \mathbb{R}^n)$

Obviously $\partial : \mathcal{D}_n(\mathbb{R}^n) \rightarrow \mathcal{D}_{n-1}(\mathbb{R}^n)$ is continuous.

Moreover, if $T_i \rightarrow T$, then $d_{H_T} \leq \liminf_{i \rightarrow \infty} d_{H_{T_i}}$.

Ex  $\xrightarrow{i \rightarrow \infty} \circ$ (cancellation).

Compactness thm (Fedderer-Fleming)

$T_i \in \mathcal{I}_n^{\text{loc}}(\mathbb{R}^n)$ st

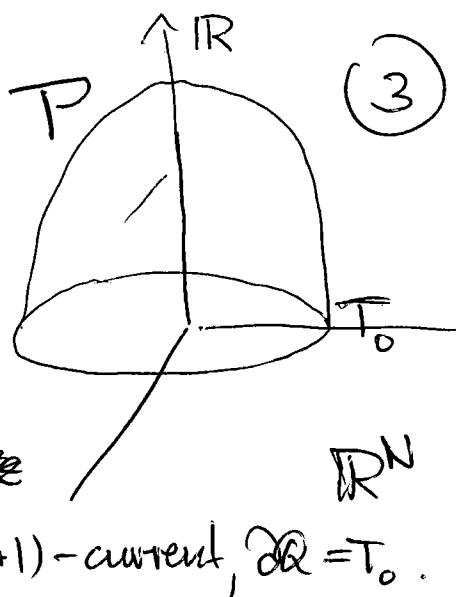
$$\sup_{i \geq 1} (M_0[T_i] + M_0[\partial T_i]) < \infty \quad \forall U \subset \mathbb{R}^n$$

Then $\exists \tilde{T} \in \mathcal{I}_n^{\text{loc}}(\mathbb{R}^n)$ st $T_{ij} \rightarrow \tilde{T}$.

Slices, Co-area formula see later.

Elliptic regularization

$T_0 \in \mathcal{I}_n(\mathbb{R}^N \times \mathbb{S}^0)$ of finite mass



$$I^\varepsilon[Q] := \frac{1}{\varepsilon} \int e^{-|z|/\varepsilon} d\mu_Q(x, z),$$

~~Q (n+1)-current, $\partial Q = T_0$.~~

Phys

Rank ~~not~~ The E.L. eqn for I^ε is

~~$$H + \frac{1}{\varepsilon} \langle e_{n+1}, r \rangle = 0,$$~~

i.e. translating soliton with speed $\frac{1}{\varepsilon}$ in the direction e_{n+1} .

Thm $\exists P^\varepsilon \in \mathcal{I}_{n+1}(\mathbb{R}^N \times \mathbb{R})$ st

~~$$\exists I^\varepsilon[P^\varepsilon] = \inf_{Q \in \mathcal{I}_{n+1}, \partial Q = T_0} I^\varepsilon[Q] \leq M[T_0]$$~~

•) $\text{spt } P^\varepsilon \subseteq \mathbb{R}^N \times [0, \infty)$

•) ~~P~~ P^ε is stationary for I^ε in $\mathbb{R}^N \times \mathbb{R} \setminus \text{spt } T_0$.

Proof Let $\mathcal{A} := \{Q \in \mathcal{I}_{n+1}(\mathbb{R}^N \times \mathbb{R}) \mid \partial Q = T_0\}$ (4)

~~Re~~ $T_0 \times [0, \infty) \in \mathcal{A}$,

$$I^\varepsilon[T_0 \times [0, \infty)] = \frac{1}{\varepsilon} \left\{ \int_{T_0}^{\infty} e^{-\frac{|z|}{\varepsilon}} dz \right\} = M[T_0] < \infty.$$

$$\Rightarrow \inf_{Q \in \mathcal{A}} I^\varepsilon[Q] \leq M[T_0] < \infty.$$

Let $P_i \in \mathcal{A}$ be a minimizing sequence:

$$I^\varepsilon[P_i] \downarrow \inf_{Q \in \mathcal{A}} I^\varepsilon[Q]$$

Federer-Fleming \Rightarrow after passing to a subsequence

$$P_i \rightarrow P, \quad \partial P = T_0, \quad d\mu_P \leq \liminf d\mu_{P_i}$$

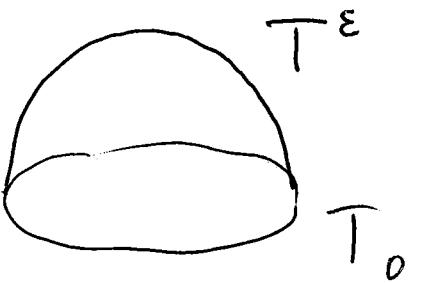
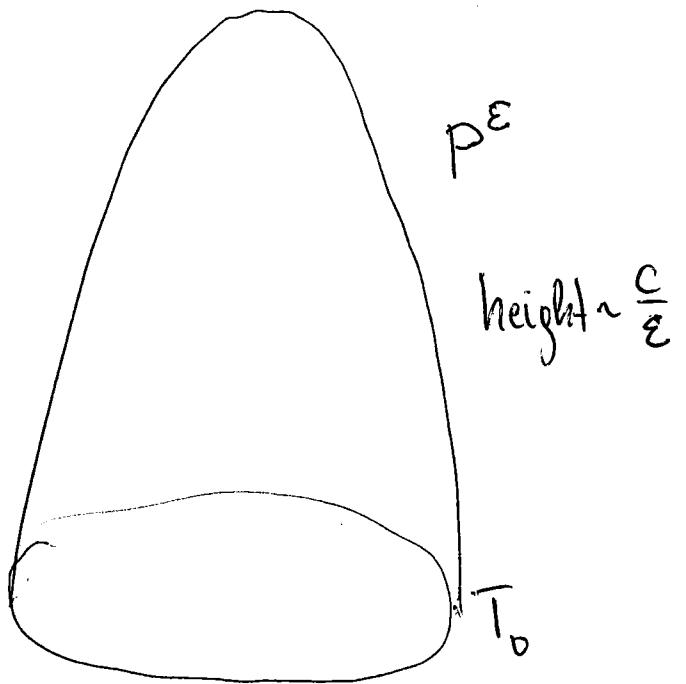
$$\text{Thus } I^\varepsilon[P] \leq \liminf_{i \rightarrow \infty} I^\varepsilon[P_i] = \inf_{Q \in \mathcal{A}} I^\varepsilon[Q]$$

$$\Rightarrow I^\varepsilon[P] = \inf_{Q \in \mathcal{A}} I^\varepsilon[Q].$$

In particular, P^ε is stationary for I^ε
in $\mathbb{R}^N \times \mathbb{R} \setminus \text{spt } T_0$

Also, if $\mu_P(\mathbb{R}^N \times (-\infty, 0)) > 0$, then $\pi^+(x, z) := (x, \max(z, 0))$ reduces $I^\varepsilon[P]$, thus $\text{spt } P^\varepsilon \subseteq \mathbb{R}^N \times [0, \infty)$

(5)



$$\gamma_{\varepsilon} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}, (x, z) \mapsto (x, \varepsilon z)$$

$$T^\varepsilon := (\gamma_\varepsilon)_\#(P^\varepsilon)$$

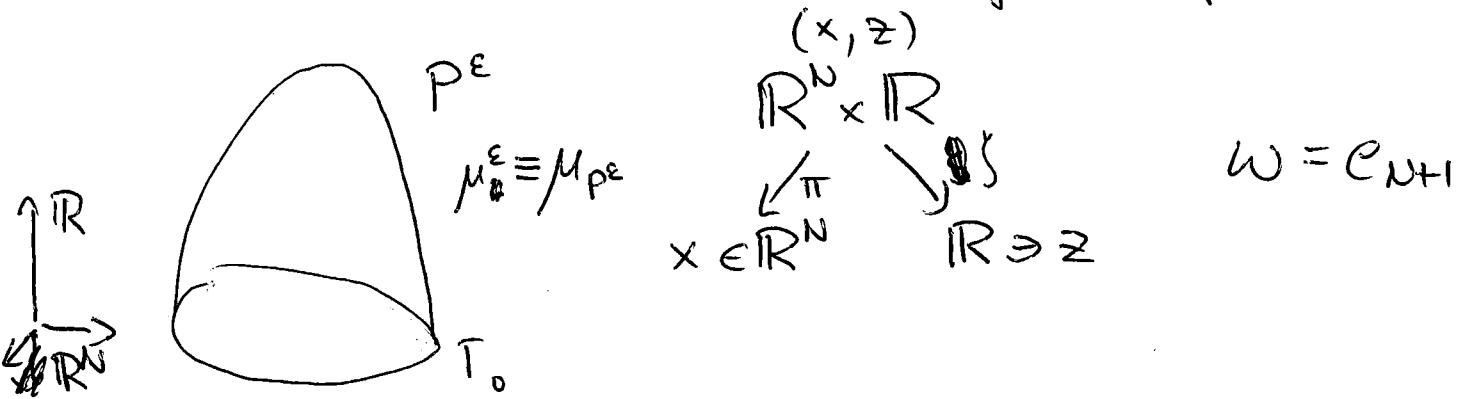
Intuition: $T^\varepsilon \approx$ space-time track of ~~T_0~~ moving by NCF.

①

The H^2 -estimate

If T_t is a smooth MCF, then $M[T_t] + \int_0^t \int_{T_s} H^2 d\mu_{T_s} ds = M[T_0]$

Would like to derive ε -version of this formula.



P^ε critical point of I^ε $\Rightarrow 0 = \int e^{-z/\varepsilon} (\xi')$

$$\Rightarrow 0 = \int e^{-z/\varepsilon} \left(\int_{R^N \times R} (\nabla I^\varepsilon) - \frac{1}{\varepsilon} \omega \cdot Y \right) d\mu^\varepsilon \quad \forall Y \in C_c^1$$

$Y = \xi(z) \omega$ (times cut-off fn)

$$\Rightarrow 0 = \int e^{-z/\varepsilon} \left(\xi' \cdot |\omega^\top|^2 - \frac{1}{\varepsilon} \xi \right) d\mu^\varepsilon$$

$$\xi' \rightarrow e^{-z/\varepsilon} \xi' \Rightarrow 0 = \int \left(\frac{1}{\varepsilon} \xi + \xi' \right) |\omega^\top|^2 - \frac{1}{\varepsilon} \xi$$

$\omega^\top + \frac{\omega^\top}{\varepsilon} = 0, \quad \omega^\top = D\xi$

$$\Rightarrow \int \xi$$

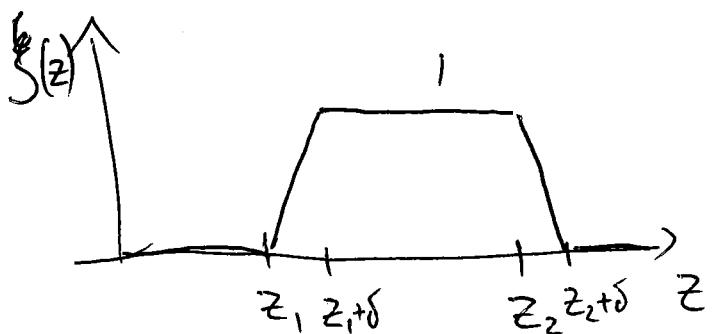
$$\xi \rightarrow e^{-\frac{z}{\varepsilon}} \xi \Rightarrow \int \frac{1}{\varepsilon} |\xi| \omega^+|^2 = \int |\xi'| \omega^+|^2 \quad (2)$$

$$\omega^+ = -\varepsilon \vec{H}, \omega^+ = D\xi$$

$$\Rightarrow \int \varepsilon |\xi| H^2 d\mu^\varepsilon = \int |\xi'| |D\xi'|^2 d\mu^\varepsilon$$

$\forall \xi: \mathbb{R} \rightarrow \mathbb{R}$

Lipschitz with
 $\text{spt } \xi \subset [0, \infty)$.



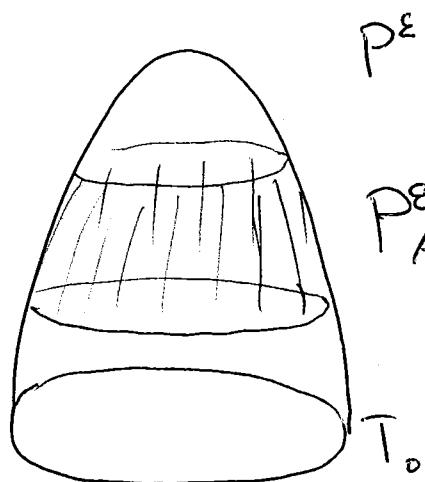
$$P(a, b) := \text{PL } \mathbb{R}^N \times (a, b)$$

$$\Rightarrow \frac{1}{\delta} \int_{P^\varepsilon(z_1, z_1 + \delta)} |D\xi|^2 d\mu^\varepsilon - \frac{1}{\delta} \int_{P^\varepsilon(z_2, z_2 + \delta)} |D\xi|^2 d\mu^\varepsilon = \int \varepsilon |\xi| H^2 d\mu^\varepsilon$$

$$\Rightarrow \lim_{\delta \rightarrow 0} \left[\int_{P^\varepsilon_{z_2}} |D\xi| d\mu_{P^\varepsilon_{z_2}} + \int_{P^\varepsilon(z_1, z_2)} \varepsilon H^2 d\mu^\varepsilon \right] = \int_{P^\varepsilon_{z_1}} |D\xi| d\mu_{P^\varepsilon_{z_1}} \leq I^\varepsilon(P) \quad (\text{coarea formula})$$

Where $P_z^\varepsilon = \text{PL } \mathbb{R}^N \times \{z\}$ is the slice at height z .

Space-time mass bound



$$P_A^\varepsilon := P^\varepsilon L_{\mathbb{R}^N \times A}, \quad A \subset \mathbb{R}$$

Thm

$M[P_A^\varepsilon] \leq (|A| + \varepsilon) \cdot I^\varepsilon[P^\varepsilon]$

Proof $| = |\omega^\top|^2 + |\omega^\perp|^2 = |D\zeta|^2 + \varepsilon^2 H^2$

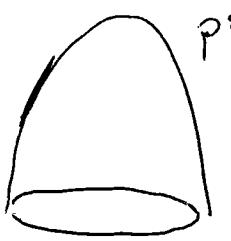
$$\begin{aligned} \Rightarrow M[P_A^\varepsilon] &= \cancel{\int_{P_A^\varepsilon} (|D\zeta|^2 + \varepsilon^2 H^2) d\mu} \\ &= \int_{P_A^\varepsilon} |D\zeta|^2 d\mu + \int_{P_A^\varepsilon} \varepsilon^2 H^2 d\mu \end{aligned}$$

$$\leq |A| I^\varepsilon[P^\varepsilon] + \varepsilon I^\varepsilon[P^\varepsilon]$$

□

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Also want estimates for T^ε

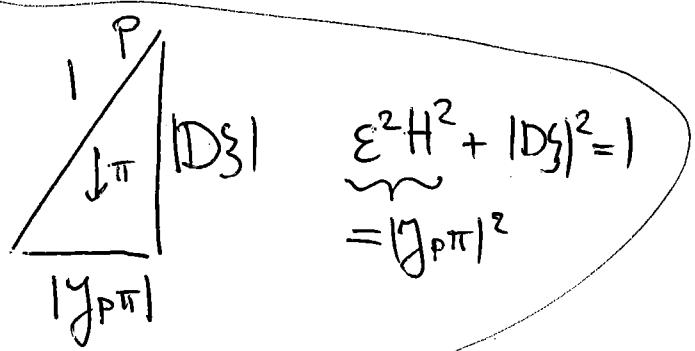


$$P_A^\varepsilon = P^\varepsilon L \mathbb{R}^N \times A$$

$$T_B^\varepsilon = T^\varepsilon L \mathbb{R}^N \times B \quad B = \varepsilon A$$

$$\mathbb{R}^N \times \mathbb{R} \xrightarrow{\pi} \mathbb{R}^N$$

$$M[\pi_\# T_B^\varepsilon] = M[\pi_\# P_A^\varepsilon] \leq \int_{P_A^\varepsilon} |J_{P^\varepsilon} \pi| d\mu$$



$$= \int_{P_A} \varepsilon H d\mu$$

$$\leq \varepsilon^2 M[P_A]^{1/2} \left(\int_{P_A} \varepsilon H^2 d\mu \right)^{1/2}$$

$$\leq \varepsilon^{1/2} (|A| + \varepsilon)^{1/2} (I^\varepsilon[P^\varepsilon])^{1/2} (I^\varepsilon[P^\varepsilon])^{1/2}$$

$$\Rightarrow \boxed{\text{Thm: } M[\pi_\# T_B^\varepsilon] \leq (FB) + \varepsilon^2) I^\varepsilon[P^\varepsilon]}$$

In particular, if we set $T_t^\varepsilon = \partial(T^\varepsilon L \mathbb{R}^N \times [t, \infty))$

$$\text{then } \text{dist}(\pi_\# T_{t+\delta}^\varepsilon, \pi_\# T_t^\varepsilon) \leq (\varepsilon^2 + \delta)^{1/2} I^\varepsilon[P^\varepsilon]$$

which will imply that $t \mapsto T_t$ is $C^{1/2}$ in the flat topology.

(5)

$$M[T_B^\varepsilon] = \int_{P_A^\varepsilon} |y_{P^\varepsilon K_\varepsilon}| d\mu \quad M_{K_\varepsilon}(x, z) = (x, \varepsilon z)$$

$$\begin{aligned} |y_{P^\varepsilon K_\varepsilon}| &= \left(\sqrt{|y_{P\pi}|^2 + \varepsilon^2 |DS|^2} \leq |y_{P\pi}| + \varepsilon |DS| \right) \\ &\leq |y_{P\pi}| + \varepsilon \end{aligned}$$

$$\Rightarrow M[T_B^\varepsilon] \leq \int_{P_A^\varepsilon} (\varepsilon + |y_{P^\varepsilon \pi}|) d\mu$$

$$\leq \varepsilon \left(\frac{|B|}{\varepsilon} + \varepsilon \right) I^\varepsilon[P^\varepsilon] + (|B| + \varepsilon^2) I^\varepsilon[P^\varepsilon]$$

$$\Rightarrow \text{Thm} \boxed{M[T_B^\varepsilon] \leq (|B| + \varepsilon^2 + (|B| + \varepsilon^2)^{\nu_2}) I^\varepsilon[P^\varepsilon]}$$

Next time: pass to limit $\varepsilon \rightarrow 0$.

will get pair $(T, \{u_t\}_{t \geq 0})$ "enhanced motion"

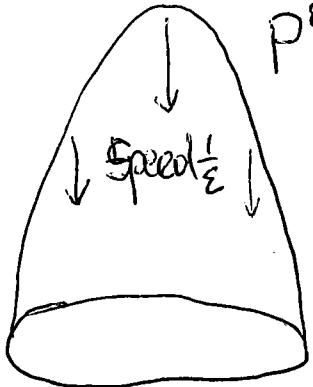
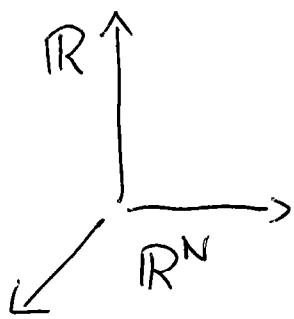


Brakke flow
current in $\mathbb{R}^N \times \mathbb{R}$ with $\partial T = T_0$

$$\begin{array}{c} M_t \geq M_{T_t} \\ \uparrow \quad \uparrow \\ \text{overflow} \quad \text{undercurrent} \end{array} \quad \text{+ Best}$$

(1)

recall

 P^ε

minimizer of

$$I^\varepsilon[\Omega] = \frac{1}{\varepsilon} \int e^{-\frac{|x-z|}{\varepsilon}} d\mu_\Omega(x, z)$$

 Ω integral $(n+1)$ -current, $\partial\Omega = T_0$. T_0 integral n -current

$$\partial T_0 = \emptyset, M[T_0] < \infty$$

in part. P^ε is stationary for I^ε in $\mathbb{R}^N \times \mathbb{R} \setminus \text{spt } T_0$,

~~$$\text{Euler eqn: } \vec{H}(x, z) + \frac{1}{\varepsilon} \omega^\perp = 0 \quad \text{for } \mu_{P^\varepsilon} \text{ a.c.}$$~~

Set $\varphi_t^\varepsilon(x, z) = (x, z - \frac{t}{\varepsilon})$, $P^\varepsilon(t) = (\varphi_t^\varepsilon)_* P^\varepsilon$, $\mu_t^\varepsilon = \mu_{P^\varepsilon(t)}$

Lemma $\{\mu_t^\varepsilon\}_{t \geq 0}$ is an integral Brakke flow

$$\text{in } W^\varepsilon := \{(x, z, t) : z > -\frac{t}{\varepsilon}, t \geq 0\}$$

Proof $\psi \in C_c^2(\mathbb{R}^N, \mathbb{R}_+)$, $\text{spt } \psi \times \{t\} \subset W^\varepsilon$

$$\begin{aligned} \frac{d}{dt} \mu_t^\varepsilon(\psi) &= \frac{d}{dt} \int \psi d(\varphi_t^\varepsilon)_*(\mu^\varepsilon) \\ &= \int (-\psi \vec{H} + \nabla_z^\perp \psi) \left(-\frac{1}{\varepsilon} \omega\right) d\mu_t^\varepsilon \end{aligned}$$

1st var. formula with $Y = \frac{\partial \varphi_t^\varepsilon}{\partial t} = -\frac{\omega}{z}$

$$\begin{aligned} &= \int \psi H^2 + \nabla \psi \cdot \vec{H} \Big) d\mu_t^\varepsilon = B(\mu_t^\varepsilon, \psi) \quad \square \end{aligned}$$

$$\omega^\perp = -H$$

$$W^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \left(\mathbb{R}^N \times \mathbb{R} \times (0, \infty) \right) \cup \left(\mathbb{R}^N \times (0, \infty) \times \{0\} \right)$$

2

~~from~~ ~~for t > 0~~

Let $W_2 = \mathbb{R}^N \times (z_1, z_2) \times [t_1, \infty)$ $t_1 \geq 0$
 If $t_1 = 0$, assume $z_1 > 0$.

For ε small, $W_2 \subset W^\varepsilon$.

For $t \geq t_1$: $\mu_t^\varepsilon (\mathbb{R}^N \times (z_1, z_2)) \leq (z_2 - z_1 + \varepsilon) M[T_0]$

Space-time mass bound from
last lecture

~~Brakke~~
 \hookrightarrow $\exists \varepsilon_i > 0, \{\bar{\mu}_t\}_{t \geq 0}^{\varepsilon_i}$ integral Brakke flow in W
 cptness claim

St $\mu_t^{\varepsilon_i} \rightarrow \bar{\mu}_t$ for all $t \geq 0$

$\bar{\mu}_t (\mathbb{R}^N \times (z_1, z_2)) \leq (z_2 - z_1) M[T_0]$

Lemma $\bar{\mu}_t$ is invariant under translations in \mathbb{Z} -direction (for all but countably many t). (3)

Proof For $\psi \in C_c^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}_+)$ let $\psi^\varepsilon(x, z) = \psi(x, z - \varepsilon)$.

Note that $\bar{\mu}_t^\varepsilon(\psi^\varepsilon) = \bar{\mu}_{t+\varepsilon\varepsilon}^\varepsilon(\psi)$

Recall that $t \mapsto \bar{\mu}_t^\varepsilon(\psi) - C(\psi)t$ is decreasing.

Fix $s > t$, send $\varepsilon \downarrow 0 \Rightarrow$

$$\bar{\mu}_t(\psi) - C(\psi)t \geq \bar{\mu}_t(\psi^\varepsilon) - C(\psi)t \geq \bar{\mu}_s(\psi) - C(\psi)s$$

Therefore $\bar{\mu}_t(\psi) \geq \bar{\mu}_t(\psi^\varepsilon) \geq \underbrace{\lim_{s \downarrow t} \bar{\mu}_s(\psi)}_{= \bar{\mu}_t(\psi)}$

$$= \bar{\mu}_t(\psi)$$

at all but countably many t

$\Rightarrow \bar{\mu}_t(\psi) = \bar{\mu}_t(\psi^\varepsilon)$ at all but countably many t □



(4)

Fix $\theta \in C_c^2((0, \infty), \mathbb{R}_+)$ st $\int \theta dz = 1$.

Define $\{\mu_t\}_{t \geq 0}$ by $\mu_t(\phi) := \bar{\mu}_t(\phi \theta)$, $\phi \in C_c^2(\mathbb{R}^N, \mathbb{R})$

\uparrow
 measure on $\mathbb{R}^N \times \mathbb{R}$
 measure on \mathbb{R}^N

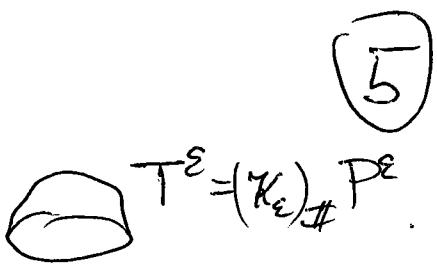
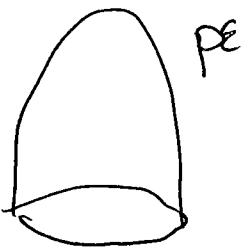
Then $\bar{\mu}_t = \mu_t \times \mathcal{L}^1$ for all but countably many $t \geq 0$

and $\{\mu_t\}_{t \geq 0}$ is an integral Brakdo flow in \mathbb{R}^N .

Indeed: \bullet) $\mathcal{B}(\bar{\mu}_t, \phi \theta) = \mathcal{B}(\mu_t, \phi)$. □

\rightarrow splits, $\bar{\mu}_t \in \mathcal{IM}_{n+1}(\mathbb{R}^N \times \mathbb{R}) \Rightarrow \mu_t \in \mathcal{IM}_n(\mathbb{R}^N)$.

Construction of T



(5)

$$\text{recall: } M[T_B^\varepsilon] \leq (|B| + \varepsilon^2 + (|B| + \varepsilon^2)^{1/2}) M[T_0]$$

$$M[\pi_\# T_B^\varepsilon] \leq (|B| + \varepsilon^2)^{1/2} M[T_0]$$

$\Rightarrow \exists \varepsilon > 0, T \in \Gamma_{n+1}(\mathbb{R}^N \times \mathbb{R})$ st

$$(1) \quad T^\varepsilon \rightarrow T, \text{ spt } T \subset \mathbb{R}^N \times [0, \infty)$$

$$(2) \quad M[T_B] \leq (|B| + |B|^{1/2}) M[T_0]$$

$$M[\pi_\# T_B] \leq (|B|)^{1/2} M[T_0]$$

Prop (1) $t \mapsto T_t := \partial(T \llcorner \mathbb{R}^N \times [t, \infty))$ is $C^{1/2}$

(2) $\mu_t \geq \mu_{T_t}$ (overflow / undercurrent)

(3) $\mu_0 = \mu_{T_0}, M[\mu_t] \leq M[\mu_0]$

(6)

Proof (1) $\text{dist}(T_t, T_{t+\delta})$

$$\leq \partial(TL_{\mathbb{R}^N \times [t, t+\delta]}) \\ \leq (\delta + \delta'^2) M[T_0]$$

(2) Want to show $\mu_t(\phi) \geq \mu_{T_t}(\phi) \quad \forall \phi \in C_c^2(\mathbb{R}^N, \mathbb{R})$

For τ fixed $T_{t+\varepsilon_i \tau}^{\varepsilon_i} \rightarrow T_t$, $\liminf_{\varepsilon_i \downarrow 0} \mu_{T_{t+\varepsilon_i \tau}}^{\varepsilon_i}(\phi) \geq \mu_{T_t}(\phi)$ (cf (1))

Then $\mu_t(\phi) = \bar{\mu}_t(\phi \theta)$

$$= \liminf_{\varepsilon_i \downarrow 0} \int \phi(x) \theta(z) d\mu_t^{\varepsilon_i}(x, z)$$

$$\geq \liminf_{\cancel{\theta}} \int \phi \theta |D\theta| d\mu_t^{\varepsilon_i}$$

$$\stackrel{\rightarrow}{=} \liminf_{\cancel{\theta}} \int \theta \int \phi d\mu_{P^{\varepsilon_i}(t)} d\mu_{\mathbb{R}^N \times [z, \infty)} dz$$

coarea formula

$$= \liminf \int \theta \int \phi d\mu_{P_{t/\varepsilon_i + z}^{\varepsilon_i}} dz$$

$$= \liminf \int \theta \int \phi d\mu_{T_{t+\varepsilon_i z}^{\varepsilon_i}} dz$$

$$\geq \int \theta \liminf \mu_{T_{t+\varepsilon_i z}^{\varepsilon_i}}(\phi) dz$$

$$\geq \mu_{T_t}(\phi)$$

$$(3) \quad \bar{\mu}_t(\mathbb{R}^N \times (z_1, z_2)) \leq (z_2 - z_1) M[\tau_0] \quad \textcircled{P}$$

$$\Rightarrow M[\mu_t] = \int d\mu_t = \int \Theta(z) d\bar{\mu}_t(x, z) \leq M[\tau_0]$$

$$\Rightarrow \mu_0 = \mu_{\tau_0} \quad \boxed{V_1}$$
$$\mu_0 \geq \mu_{\tau_0}$$