Exterior derivative & Stokes theorem

Computationally, if $\omega = \sum \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$

then $d\omega = \sum d\omega_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$

Ex For differential forms in $\mathbb{R}^3$ we have

(i) $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

(ii) $d(P dx + Q dy + R dz) = dP \wedge dx + dQ \wedge dy + dR \wedge dz$

$$= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz\right) \wedge dx$$
$$+ \left(\frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial y} dy + \frac{\partial Q}{\partial z} dz\right) \wedge dy$$
$$+ \left(\frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \frac{\partial R}{\partial z} dz\right) \wedge dz$$
$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx \wedge dz + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz$$

(iii) $d(u\, dy \wedge dz + v\, dz \wedge dx + w\, dx \wedge dy)$

$$= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx \wedge dy \wedge dz$$

So this fits together into the commutative diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}^3) \arrow[r, "\text{grad}"] \arrow[d, "Id"'] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\text{curl}"] \arrow[d, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[r, "\text{div}"] \arrow[d, "\beta"] & C^\infty(\mathbb{R}^3) \arrow[d, "*"] \\
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3)
\end{tikzcd}$$

where the vertical isomorphisms are defined as

$$\flat\left(P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}\right) = P\,dx + Q\,dy + R\,dz$$

$$\beta(X) = i_X(dx\wedge dy\wedge dz) = (dx\wedge dy\wedge dz)(X,\cdot,\cdot)$$

$$*(f) = f\,dx\wedge dy\wedge dz$$

(In more detail $\beta\left(A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C\frac{\partial}{\partial z}\right) = A\,dy\wedge dz + B\,dz\wedge dx + C\,dy\wedge dz$)

In particular $\text{curl}\circ\text{grad} = 0$ & $\text{div}\circ\text{curl} = 0$ becomes $\boxed{d\circ d = 0}$

Goal: Generalize this to mfds & arbitrary dim.

Thm (exterior derivative)

There are unique linear operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$, $\forall k$, such that

(i) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ if $\omega \in \Omega^k(M)$, $\eta \in \Omega^\ell(M)$

(ii) $d \circ d = 0$

(iii) $df(X) = X(f)$ if $f \in \Omega^0(M) = C^\infty(M)$, $X \in \mathfrak{X}(M)$

Moreover, if $F : M \to N$ is smooth, then

$$F^*(d\omega) = d(F^*\omega) \qquad \omega \in \Omega^k(N).$$

In particular, if $\omega = \sum \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$

then $d\omega = \sum d\omega_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

Proof: First establish locality: If ω_1 and ω_2 agree on $U \subset M$ open, then $d\omega_1 = d\omega_2$ on U.

Indeed $p \in U$, $\psi \in C^\infty_c(U)$ st $\psi \equiv 1$ near p.

Then $\eta := \omega_1 - \omega_2$ satisfies $\psi\eta \equiv 0$, so

evaluating $0 = d(\psi\eta) = d\psi \wedge \eta + \psi\, d\eta$ at p

we get $d\eta_p = 0$, i.e. $d\omega_1 = d\omega_2$ on U.

Uniqueness: Given $\omega \in \Omega^k(M)$, $p \in M$ in a (coordinate) nbd U

can write $\omega = \sum \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$.

Chose $\tilde{\omega}_{i_1 \dots i_k}$, $\tilde{x}^i$ with cpt support in U,

that agree with $\omega_{i_1 \dots i_k}$, x^i near p

Locality $\Rightarrow$ $d\omega_p = \sum d(\tilde{\omega}_{i_1 \dots i_k})_p \wedge d\tilde{x}^{i_1} \dots \wedge d\tilde{x}^{i_k}$

$\Rightarrow$ uniqueness.

Existence: By locality, enough to construct

$d : \Omega^k(U) \to \Omega^{k+1}(U)$ for $U \subset \mathbb{R}^n$ (or $\mathbb{H}^n$) open

that satisfies (i) - (iii) & commutes with pullback.

So for $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^k(U)$

we define $d\omega := \sum_{i_1 < \dots < i_k} d\omega_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Omega^{k+1}(U)$

and check that this is clearly linear and satisfies:

(i) If $\omega = u\, dx^{i_1} \wedge \dots \wedge dx^{i_k}$, $\eta = v\, dx^{j_1} \wedge \dots \wedge dx^{j_l}$

then $d(\omega \wedge \eta) = d(uv\, dx^{i_1} \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l})$

$= \underbrace{d(uv)}_{= v\,du + u\,dv} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l}$

$= (du \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}) \wedge (v\, dx^{j_1} \wedge \dots \wedge dx^{j_l})$
$+ (-1)^k (u\, dx^{i_1} \wedge \dots \wedge dx^{i_k}) \wedge (dv \wedge dx^{j_1} \wedge \dots \wedge dx^{j_l})$.

(ii) $$d(df) = d\left(\sum_j \frac{\partial f}{\partial x^j} dx^j\right)$$

$$= \sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j$$

$$= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = 0$$

So $$d(d\omega) = \sum_{i_1 < \dots i_k} \underbrace{dd\omega_{i_1, \dots i_k}}_{=0} \wedge dx^{i_1} \dots \wedge dx^{i_k} = 0 .$$

(iii) $$df(X) = \frac{\partial f}{\partial x^i} dx^i \left(X^j \frac{\partial}{\partial x^j}\right) = X^i \frac{\partial f}{\partial x^i} = X(f) .$$

And finally for $F: V \to U$ smooth

$$F^*(d(u\, dx^{i_1} \wedge \dots \wedge dx^{i_k})) = F^*(du \wedge dx^{i_1} \dots \wedge dx^{i_k})$$

$$= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \dots \wedge d(x^{i_k} \circ F)$$

and $$d(F^*(u\, dx^{i_1} \wedge \dots \wedge dx^{i_k})) = d((u \circ F)\, d(x^{i_1} \circ F) \wedge \dots \wedge d(x^{i_k} \circ F))$$

$$= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \dots \wedge d(x^{i_k} \circ F)$$

$\boxed{/\!/}$

In general, if $A = \bigoplus_k A^k$ is a graded algebra, then a <u>derivation of degree d</u> is a linear map $D : A \to A$, st $D(A^k) \subseteq A^{k+d}$ and

$$D(xy) = (Dx)y + (-1)^{kd} x(Dy) \quad , \text{ if } x \in A^k, y \in A^\ell .$$

So the theorem can be summarized by saying:

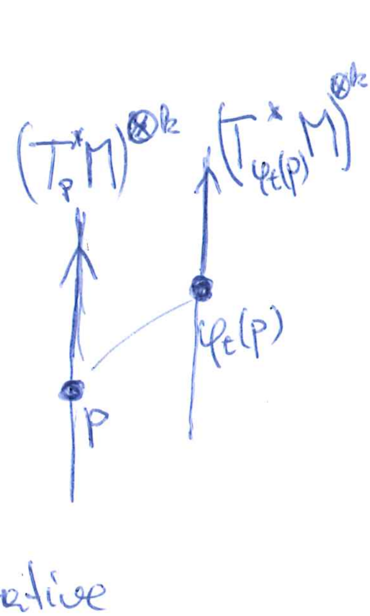
"The differential on functions ~~extends a~~ extends uniquely to a derivation of degree $+1$ of $\Omega^*(M)$ whose square is zero."

<u>Note:</u> Interior multiplication

$$i_X : \Omega^*(M) \to \Omega^*(M), \quad i_X\omega = \omega(X, \dots, \dots)$$

is a derivation of degree -1 whose square is zero.

They are related by:



<u>Thm</u> (Cartan's magic formula)

$$\boxed{\mathcal{L}_X = i_X \circ d + d \circ i_X}$$

Here, $\mathcal{L}_X \omega := \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* \omega$ is the Lie derivative (φ_t is the flow of X)

<u>Proof</u> By definition

$$(\mathcal{L}_X \omega)_p = \frac{d}{dt}\Big|_{t=0} (\varphi_t^* \omega)_p \in \Lambda^k T_p^* M$$

So $\mathcal{L}_X(\omega \wedge \eta) = \frac{d}{dt}\Big|_{t=0} (\varphi_t^* \omega \wedge \varphi_t^* \eta)$

$$= \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta ,$$

i.e. $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ is a derivation of degree 0.

Also note that $\mathcal{L}_X f = \frac{d}{dt}\Big|_{t=0} f \circ \varphi_t = X(f)$.

Now: $\left.\begin{array}{l} i_X \text{ der. of degree } -1 \\ d \text{ der. of degree } +1 \end{array}\right\} \underset{\text{(just algebra)}}{\Longrightarrow}$ $i_X \circ d + d \circ i_X$ is a derivation of degree 0.

Also $(i_X d + d i_X) f = i_X df = df(X) = X(f)$.

Moreover: $\varphi_t^* d = d \varphi_t^* \Rightarrow \mathcal{L}_X d = d \mathcal{L}_X$

and $d(d i_X + i_X d) = d i_X d = (d i_X + i_X d) d$.

So $\mathcal{L}_X$ & $i_X \circ d + d \circ i_X$ are both degree 0 derivations that agree on functions & commute with d. Hence they are equal

Stokes thm

M^n oriented smooth mfd with bdry

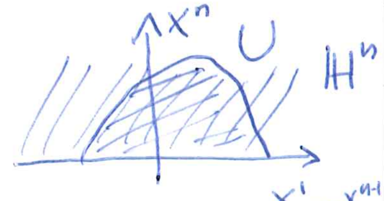
$$\Rightarrow \int_M d\omega = \int_{\partial M} \omega \qquad \forall \omega \in \Omega_c^{n-1}(M).$$

Proof: Enough to show $\int_U d\omega = \int_{\partial U} \omega$ for $U \subset \mathbb{H}^n$ open, $\omega \in \Omega_c^{n-1}(U)$.

So consider $\omega = \sum_{i=1}^n \omega_i \, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \in \Omega_c^{n-1}(U)$

(hat means dx^i omitted)

Then $d\omega = \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \ldots \wedge dx^n$



Thus $\int_U d\omega = \sum_{i=1}^n (-1)^{i-1} \int_{\mathbb{H}^n} \frac{\partial \omega_i}{\partial x^i} dx^1 \ldots dx^n$

$$\underset{\text{FTC}}{=} (-1)^{n-1} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underbrace{\int_0^{\infty} \frac{\partial \omega_n}{\partial x^n} dx^n}_{= -\omega_n(x^1, \ldots, x^{n-1}, 0)} \ldots dx^1$$

$$= (-1)^n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \omega_n(x^1, \ldots, x^{n-1}, 0) \, dx^1 \ldots dx^{n-1}$$

$$= \int_{\partial U} \omega$$

since $dx^n|_{\partial \mathbb{H}^n} = 0$ and $(x^1, \ldots, x^{n-1})$ pos. or. $\Leftrightarrow$ n even. $\square$

Low dimensional cases:

(1) $f(b) - f(a) = \int_a^b f'(x)\,dx$ (FTC)

(2) $D \subset \mathbb{R}^2$: $\int_{\partial D} P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ (Green)

(3) In $\mathbb{R}^3$ remembering our grad, curl, div diagram we get

(i) $f(\gamma(b)) - f(\gamma(a)) = \int_\gamma \operatorname{grad} f \cdot \dot{\gamma}(t)\,dt$

(ii) $\int_{\partial S} (f_1\,dx + f_2\,dy + f_3\,dz) = \int_S \operatorname{curl} F \cdot n\,dA$

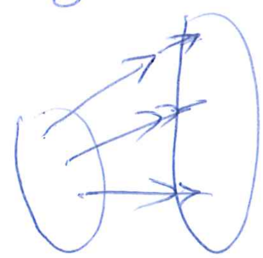
(iii) $\int_{\partial D} F \cdot n\,dA = \int_D \operatorname{div} F\,dV$

where $F = (f_1, f_2, f_3)$ & n is the outwards unit normal.

In $\mathbb{R}^n$ have volume form $dV = dx^1 \wedge \ldots \wedge dx^n$

For $X \in \mathcal{X}(\mathbb{R}^n)$ define $\operatorname{div} X$ by

$$\mathcal{L}_X dV = (\operatorname{div} X)\, dV$$



(how volume expands along the flow of X)

Cartan's magic formula $\Rightarrow \mathcal{L}_X dV = d(i_X dV)$

Now $$i_X dV = (dx^1 \wedge \ldots \wedge dx^n)\left(\sum_i X^i \frac{\partial}{\partial x^i}\right)$$

$$= \sum_i (-1)^{i-1} X^i dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n$$

So $$d(i_X dV) = \sum_i \frac{\partial X^i}{\partial x^i} dx^1 \wedge \ldots \wedge dx^n,$$

namely $$\operatorname{div} X = \sum_{i=1}^n \frac{\partial X^i}{\partial x^i}.$$

And Stokes gives

$$\int_D \operatorname{div} X \, dV = \int_{\partial D} i_X dV \equiv \int_{\partial D} X \cdot n \, dA$$

(divergence theorem)

Rmk: Also works on Riemannian mflds.

Applications of Stokes theorem

<u>Prop</u> M smooth compact mfd with bdry $\partial M \neq \emptyset$

$\Rightarrow \nexists\, r : M \to \partial M$ st $r|_{\partial M} = \mathrm{id}$.

(<u>Recall:</u> Prop $\Rightarrow$ Brouwers FPT)

<u>Proof:</u> Can assume M orientable (if not lift to double cover)

Choose positive $\lambda \in \Omega^{n-1}(\partial M)$

If $\exists$ such r, then $0 = \int_M r^* \underbrace{d\lambda}_{=0} = \int_M d r^* \lambda = \int_{\partial M} \lambda > 0$ ↯ ☒

Stokes

<u>Thm</u> (degree formula)

$f : M \to N$ smooth map between closed oriented connected smooth n-manifolds. Then:

$$\boxed{\int_M f^* \omega = \deg(f) \int_N \omega} \qquad \forall \omega \in \Omega^n(N).$$

<u>Note:</u> •) degree formula $\Rightarrow$ hairy ball thm

•) have already seen degree formula in special case where f is orientation pres/rev. diffeo.

Lemma: If $f: M \to N$ extends smoothly to $F: X \to N$, where $\partial X = M$, then

$$\int_M f^* \omega = 0 \quad \forall \omega \in \Omega^n(N).$$

Proof: $$\int_M f^*\omega = \int_{\partial X} F^*\omega \underset{\text{Stokes}}{=} \int_X F^* \underbrace{d\omega}_{=0} = 0. \quad \boxed{}$$

Cor: If $f_0, f_1 : M \to N$ are homotopic, then

$$\int_M f_0^* \omega = \int_M f_1^* \omega \quad \forall \omega \in \Omega^n(N)$$

Proof: Let $F: I \times M \to N$ be a smooth homotopy.

$$\text{Lemma} \Rightarrow 0 = \int_{\partial(I\times M)} (\partial F)^* \omega = \int_M f_1^* \omega - \int_M f_0^* \omega. \quad \boxed{}$$

$\partial(I \times M) = \{1\} \times M - \{0\} \times M$

Note: This is reminiscent of the lemmas/arguments from the lecture on integer degree theory (cf. Milnor's book on difftopo)

Proof of the degree thm

Let y be a regular value for $f: M \to N$ ($\exists y$ by Sard)

Let $V \ni y$ open st $f^{-1}(V) = \underbrace{U_1 \cup \dots \cup U_k}_{\text{open disjoint}}$, st $f|_{U_i}: U_i \xrightarrow{\cong} V$.

If $\omega \in \Omega^n(N)$ satisfies $\mathrm{spt}(\omega) \subset V$, then

$$\int_M f^*\omega = \sum_i \int_{U_i} f^*\omega = \sum_i \sigma_i \int_V \omega ,$$

where $\sigma_i = \begin{cases} +1 & f: U_i \to V \text{ orient. preserving} \\ -1 & \text{— '' — reversing} \end{cases}$

So $\displaystyle\int_M f^*\omega \underset{(*)}{=} \deg(f) \int_N \omega \quad \forall \omega \in \Omega^n(N)$ with $\mathrm{spt}(\omega) \subset V$.

Isotopy Lemma: $\forall z \in N \; \exists h: N \to N$ diffeo isot. to id that carries y to z

Cptness $\Rightarrow N = h_1(V) \cup \dots \cup h_n(V)$ for such h_i.

partition of unity $\Rightarrow$ suffices to prove

that $\displaystyle\int_M f^*\omega = \deg(f) \int_N \omega \quad \forall \omega \in \Omega^n(N)$ with $\mathrm{spt}(\omega) \subset h(V)$

Cor $\Rightarrow \displaystyle\int_M f^*\omega = \int_M (h \circ f)^*\omega = \int_M f^* \underbrace{h^*\omega}_{\text{supported in } V} \underset{(*)}{=} \deg(f) \int_N h^*\omega = \deg(f) \int_N \omega.$ $\blacksquare$

Rmk More generally, if $X, Y \subset M$ are submfds of complementary dim, then

$$X \cdot Y = \int_M \eta_X \wedge \eta_Y \quad \text{(intersection number)}$$

Here, for $Z^k \subset M^n$ submfd the Poincare dual $[\eta_Z]$ is determined by $\int_Z \omega = \int_M \eta_Z \wedge \omega \quad \forall \omega \in \Omega^k(M)$.

Q: Let $\omega \in \Omega^n(S^n)$. When is $\omega = d\lambda$ for some $\lambda \in \Omega^{n-1}(S^n)$?

Note: If $\omega = d\lambda$, then $\int_{S^n} \omega \underset{\text{Stokes}}{=} 0$.

On the other hand, we have

$$\dim\left(\Omega^n(S^n) / \operatorname{image}\left(d : \Omega^{n-1}(S^n) \to \Omega^n(S^n)\right)\right) = 1$$

(for $n = 1$ this was HW, for $n \geq 2$ we will prove it later via Mayer - Vietoris)

So: $\boxed{\omega = d\lambda \text{ for some } \lambda \iff \int_{S^n} \omega = 0}$

In particular, via stereographic projection for $\omega \in \Omega^n_c(\mathbb{R}^n)$ this yields

$$\boxed{\omega = d\lambda \text{ for some } \lambda \in \Omega^{n-1}_c(\mathbb{R}^n) \Leftrightarrow \int_{\mathbb{R}^n} \omega = 0} \quad (*)$$

Prop Let M be an oriented smooth connected cpt n-mfd. Then for $\omega \in \Omega^n(M)$ we have

$$\omega = d\lambda \text{ for some } \lambda \in \Omega^{n-1}(M) \Leftrightarrow \int_M \omega = 0.$$

Proof "$\Rightarrow$" follows directly from Stokes.

"$\Leftarrow$" Fix $V \subset M$ open diffeomorphic to $\mathbb{R}^n$, and $\omega_1 \in \Omega^n(M)$ with $\operatorname{spt}(\omega_1) \subset V$ and $\int_M \omega_1 = 1$.

Write $\omega \sim \omega' :\Leftrightarrow \exists \lambda \in \Omega^{n-1}(M) : \omega - \omega' = d\lambda$.

By $(*)$ for every $\omega \in \Omega^n(M)$ with $\operatorname{spt}(\omega) \subset V$ we get $\omega \sim c\,\omega_1$, where $c = \int_M \omega$.

In general, as before $M = h_1(V) \cup \ldots \cup h_n(V)$, where h_i are diffeos isotopic to id.

Since "$\sim$" is homotopy invariant (see next week) for any $\omega \in \Omega^n(M)$ we get $\omega \sim c\,\omega_1$, where $c = \int_M \omega$

In particular, $\omega \sim 0 \Leftrightarrow \int_M \omega = 0$. ☒