Vector bundles

Idea: Put vector spaces $\{E_p\}_{p\in M}$ together into global structure $E = \coprod_p E_p$.

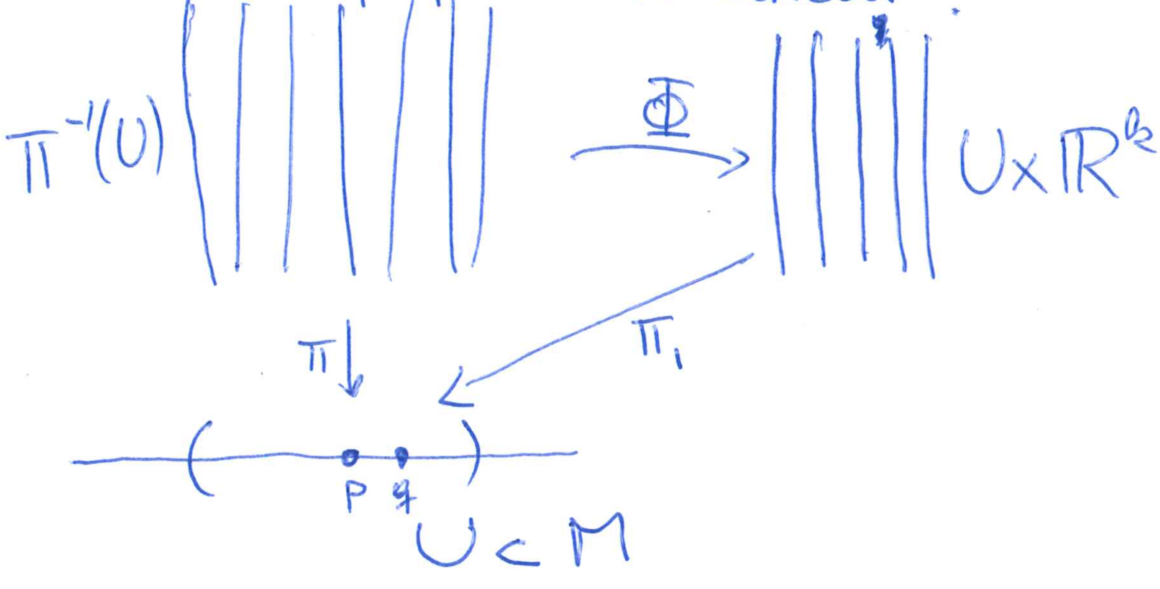
Def: A *smooth vector bundle* over a smooth mfd M is a smooth mfd E together with a smooth surjection $\pi : E \to M$, st:

(i) $\forall p \in M$ the fibre $E_p = \pi^{-1}(p)$ has the structure of a k-dim vector space

(ii) $\forall p \in M$ $\exists U \ni p$ open & $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ diffeo, st. $\pi_1 \circ \Phi = \pi$ and $\forall q \in U$ the map

$$\Phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$$

is linear.



Examples:

-) <u>tangent bundle</u> $TM = \coprod_p T_pM$

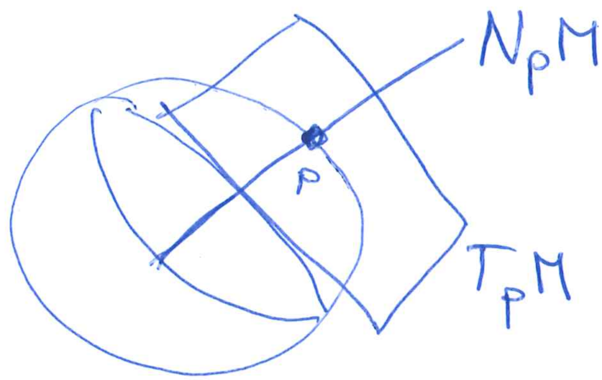
-) <u>cotangent bundle</u> $T^*M = \coprod_p T_p^*M$

-) If $M^m \subset \mathbb{R}^n$ also get <u>normal bundle</u>

  $$NM = \coprod_p (T_pM)^\perp .$$

  So $T\mathbb{R}^n\big|_M = TM \oplus NM$, namely

  $\forall p \in M$ we have $\mathbb{R}^n \cong T_p\mathbb{R}^n = \underbrace{T_pM}_{m\text{-dim}} \oplus \underbrace{N_pM}_{(n-m)\text{-dim}}$



Terminology:

-) E is called the <u>total space</u>, π the <u>bundle projection</u>

-) Φ is called a <u>local trivialization</u>

-) $k = \dim E_p = \dim E - \dim M$

  is called the <u>rank</u> of the vector bundle

__Def:__ A __section__ of $\pi : E \to M$ is a smooth function $\sigma : M \to E$, s.t. $\pi \circ \sigma = id_M$.

__Notation:__ $\Gamma(E) := \{\sigma \mid \sigma \text{ section of } E \xrightarrow{\pi} M\}$

__Ex:__ ·) sections of the trivial line bundle $M \times \mathbb{R} \to M$ are simply smooth functions on M.

·) sections of TM are vector fields $X = X^i \frac{\partial}{\partial x^i}$.

·) sections of T^*M are covector fields $\omega = \omega_i dx^i$.

__Note:__ If $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ & $\Phi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{R}^k$ are local trivializations, then

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^k$$

has the form $\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p) v)$,

where $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(\mathbb{R}, k)$ is smooth.

So vector bundles can be described in terms of the transition functions $\tau_{\alpha\beta}$.

How to build more vector bundles?

$E \to M$, $F \to M$ two vector bundles.

<u>sum bundle</u> $E \oplus F := \coprod_p E_p \oplus F_p$

with transition maps $\begin{pmatrix} \tau^E & 0 \\ 0 & \tau^F \end{pmatrix}$

<u>product bundle</u> $E \otimes F := \coprod_p E_p \otimes F_p$

with transition maps $\tau^E \otimes \tau^F$

<u>Note:</u> $\mathrm{rk}(E \oplus F) = \mathrm{rk}(E) + \mathrm{rk}(F)$

$\mathrm{rk}(E \otimes F) = \mathrm{rk}(E) \cdot \mathrm{rk}(F)$

Similarly, we can build

<u>dual bundle</u> E^*

<u>(k,l)-tensor bundle</u> $T^{k,l} E^* = \underbrace{E^* \otimes \ldots \otimes E^*}_{k\text{-times}} \otimes \underbrace{E \otimes \ldots \otimes E}_{l\text{-times}}$

<u>alternating k-tensor bundle</u> $\Lambda^k E^* \subset T^{k,0} E^*$

<u>Note:</u> $\overset{\mathrm{rk}}{\cancel{\dim}}\, E^* = \overset{\mathrm{rk}}{\cancel{\dim}}\, E$

$\overset{\mathrm{rk}}{\cancel{\dim}}\, T^{k,l} E^* = (\overset{\mathrm{rk}}{\cancel{\dim}}\, E)^{k+l}$

$\overset{\mathrm{rk}}{\cancel{\dim}}\, \Lambda^k E^* = \binom{\overset{\mathrm{rk}}{\cancel{\dim}}\, E}{k}$

<u>Def</u>: A <u>bundle homomorphism</u> is a smooth map $F: E \to E'$ that covers some $f: M \to M'$, namely

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

commutes, such that $\forall p \in M$: $F|_{E_p} : E_p \to E'_{f(p)}$ is linear.

<u>Ex</u>: •)
$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

•)
$$\begin{tikzcd} E|_S \arrow[r, hook] \arrow[d] & E \arrow[d] \\ S \arrow[r, hook] & M \end{tikzcd}$$

<u>Def</u>: A <u>bundle homomorphism over M</u> is a bundle homomorphism that covers id_M.

<u>Ex</u>: $TS^n \underset{\uparrow}{\cong} S^n \times \mathbb{R}^n \iff n = 1, 3, 7$.

$\exists$ bundle isomorphism over S^n.