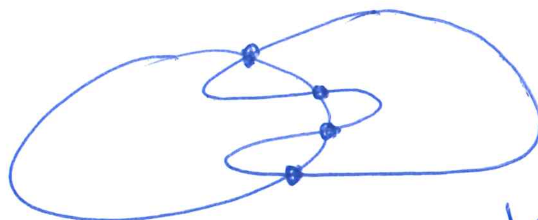
Mod 2 degree & applications

Idea: $X \pitchfork Y$ submfds of complementary dim (i.e. $\dim X + \dim Y = n$)

$\Rightarrow$ intersection points appear/disappear in pairs:



X $\qquad$ Y

In particular, for $f : M^n \longrightarrow N^n$, taking $X = M$, $Y = \{y\}$

cpt mfds of same dim

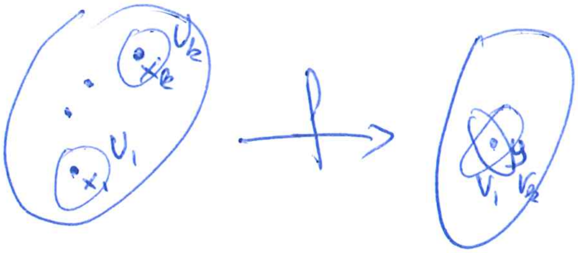
hope to get a homotopy invariant count $\# f^{-1}(y)$ (mod 2).

Preliminary observation: $\# f^{-1}$ is locally constant,

i.e. $\forall\, y \in N$ regular value $\exists\, V \ni y$ open nbd st:

$$\# f^{-1}(y) = \# f^{-1}(y') \qquad \forall y' \in V$$

Indeed: $f^{-1}(y) = \{x_1, \dots, x_k\}$, $\exists\, U_i \ni x_i$ open nbd,



M $\qquad$ N

st $f|_{U_i} : U_i \longrightarrow V_i$ diffeo, where $V_i = f(U_i) \subset N$ open.

May take

$$V = \bigcap_{i=1}^{k} V_i \setminus f\Big(M \setminus \bigcup_{i=1}^{k} U_i\Big).$$

Homotopy Lemma

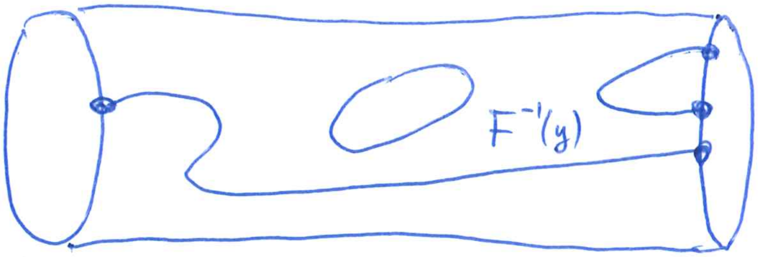
$f, g : M^n \to N^n$ smoothly homotopic
(M^n: cpt without bdry)

y regular value for f & g $\Rightarrow \# f^{-1}(y) \equiv \# g^{-1}(y) \pmod 2$.

<u>Proof:</u> By assumption $\exists F : M \times [0,1] \to N$ smooth, st. $F(x,0) = f(x)$ & $F(x,1) = g(x)$.

Suppose first y is also a regular value for F.
Then, $F^{-1}(y)$ is a cpt 1-dim mfd with bdry

$$\partial F^{-1}(y) = F^{-1}(y) \cap (M \times \{0\} \cup M \times \{1\}) = f^{-1}(y) \times \{0\} \cup g^{-1}(y) \times \{1\}$$



$M \times \{0\}$ $\qquad$ $M \times [0,1]$ $\qquad$ $M \times \{1\}$

Thus, $\# \partial F^{-1}(y) = \# f^{-1}(y) + \# g^{-1}(y)$ is even. $\Rightarrow \# f^{-1}(y) \equiv \# g^{-1}(y) \pmod 2$.

Suppose now y is not a regular value for F.
y reg. value for $f, g \Rightarrow \exists V \ni y$ open nbd consisting of regular values st:

$$\# f^{-1}(y) = \# f^{-1}(y') \ \& \ \# g^{-1}(y) = \# g^{-1}(y') \quad \forall y' \in V.$$

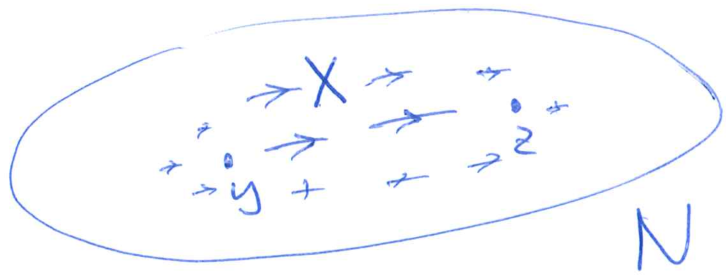
Sard $\Rightarrow \exists y' \in V$ that is also regular for F.

$$\Rightarrow \# f^{-1}(y) \equiv \# f^{-1}(y') \equiv \# g^{-1}(y') \equiv \# g^{-1}(y) \pmod 2$$

$\blacksquare$

<u>Homogeneity lemma</u> N smooth connected mfld.

Then $\forall y, z \in \mathrm{Int}(N)$, $\exists$ diffeo $h: N \to N$ that is smoothly isotopic to the identity and maps y to z.



<u>Def</u>: Diffeos $f_0, f_1 : M \to N$ are smoothly isotopic if $\exists F: M \times [0,1] \to N$ smooth homotopy, st.
$\forall t \in [0,1]$ the map $f_t(x) = F(x,t)$ is a diffeo from M to N.

Indeed, the homogeneity lemma has been proved in problem set #2, by constructing a suitable vector field X, and setting $h = \varphi_1$,

$$\text{where} \begin{cases} \frac{d}{dt}\varphi_t = X \circ \varphi_t \\ \varphi_0 = id_N \end{cases}$$

Thm (mod 2 degree)

Let M, N be smooth mfds of the same dimension,
cpt without bdry   connected

Then, to any continuous $f: M \to N$ we can associate $\deg_2(f) \in \mathbb{Z}_2$, such that:

(i) If f is smooth, then $\deg_2(f) \equiv \# f^{-1}(y)$ for any regular value $y \in N$.

(ii) If $f, g: M \to N$ are homotopic, then $\deg_2(f) = \deg_2(g)$.

Proof: If f is smooth, we define
$$\deg_2(f) :\equiv \# f^{-1}(y), \text{ where } y \in N \text{ is any regular value.}$$
If $z \in N$ is any other regular value, choose $h: N \to N$ diffeo isotopic to id, s.t. $h(y) = z$.

Homotopy lemma $\Rightarrow \#(h \circ f)^{-1}(z) \equiv \# f^{-1}(z)$.

But $(h \circ f)^{-1}(z) = f^{-1}(y)$, thus $\# f^{-1}(y) \equiv \# f^{-1}(z)$,

so $\deg_2(f)$ is well-defined, i.e. independent of the choice of regular value.

Now suppose $f, g : M \to N$ are smoothly homotopic.

Sard $\Rightarrow$ $\exists\, y \in N$ regular for both f & g.

$$\Rightarrow \deg_2(f) \equiv \# f^{-1}(y) \underset{\substack{\uparrow \\ \text{Homotopy Lemma}}}{\equiv} \# g^{-1}(y) \equiv \deg_2(g).$$

Finally, by approximation for any $f : M \to N$ continuous we can find smooth $\tilde{f} : M \to N$ homotopic to f, and set $\deg_2(f) := \deg_2(\tilde{f})$.

If $\tilde{f}_0, \tilde{f}_1 : M \to N$ are smooth, and $\tilde{f}_0$ is homotopic to $\tilde{f}_1$, then $\tilde{f}_0$ is smoothly homotopic to $\tilde{f}_1$.
So this is well-defined & satisfies (i)&(ii). $\square$

<u>Ex</u> M cpt smooth mfd without bdry.

Then ~~id_M has~~ $1 = \deg_2(id_M) \neq \deg_2(c) = 0$,

so the identity map is not homotopic to a constant map.

Applications

$f : M^n \to \mathbb{R}^{n+1}$ smooth
$\quad\uparrow$ cpt without bdry

For $z \in \mathbb{R}^{n+1} \setminus f(M^n)$ define $w_{f,z} : M^n \to S^n$
$$x \longmapsto \frac{f(x) - z}{|f(x) - z|}$$

Def: $W_2(f,z) := \deg_2(w_{f,z})$ is called the <u>mod 2 winding number</u> of f around z.

Take $M^n = S^n$.

<u>Easy observation:</u> If $0 \notin f(S^n)$ and $\underbrace{f(-x) = f(x)}_{\text{"even"}}$, then $W_2(f,0) = 0$.

<u>Indeed:</u> Considering $f_t(x) = \dfrac{f(x)}{(1-t) + t|f(x)|}$

Can reduce to $f : S^n \to S^n$. Then $W_2(f,0) = \deg_2(f)$

f even $\Rightarrow$
$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "\pi"'] & S^n \\ \mathbb{RP}^n \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd}$$

Since π is a $2:1$ cover, conclude that

$$\deg_2(f) \equiv 2 \deg_2(\bar{f}) \equiv 0 . \quad \square$$

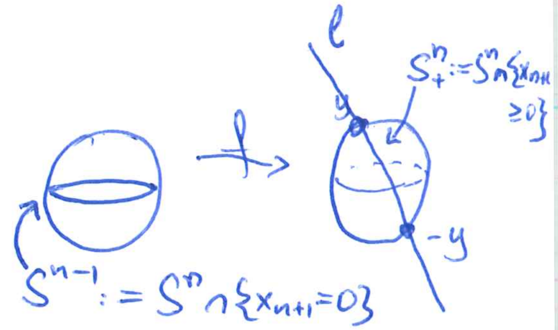
Thm (Borsuk-Ulam)

If $f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ is odd (i.e. $f(-x) = -f(x)$), then $W_2(f, 0) = 1$.

Proof: May assume $f : S^n \to S^n$, so $w_{f,0} = f$.

$n=0$: $f = \pm \mathrm{id} \Rightarrow \deg_2(f) = 1$.

$n-1 \rightsquigarrow n$: Given odd smooth $f : S^n \to S^n$, define $g := f|_{S^{n-1}} : S^{n-1} \to S^n$



Sard $\Rightarrow \exists y \in \mathrm{Int}(S^n_+)$ regular for f & g.

Note: y regular for $g \Leftrightarrow g(S^{n-1}) \cap \ell = \phi$, where $\ell := \mathbb{R}y$.

Symmetry $\Rightarrow \deg_2(f) \equiv \frac{1}{2} \# f^{-1}(\ell) \equiv \# f_+^{-1}(\ell)$,

where $f_+ := f|_{S^n_+}$.

Let $\pi : \mathbb{R}^{n+1} \to \ell^\perp \cong \mathbb{R}^n$ orthogonal projection.

Then $\pi \circ g : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ odd,

so by induction $W_2(\pi \circ g, 0) = 1$.

Finally, $W_2(\pi \circ g, 0) = \#(\pi \circ f_+)^{-1}(0) = \# f_+^{-1}(\ell)$ ▨

Exer

Cor (Borsuk-Ulam)

$g : S^n \to \mathbb{R}^n$ continuous $\Rightarrow \exists x \in S^n : g(-x) = g(x)$.

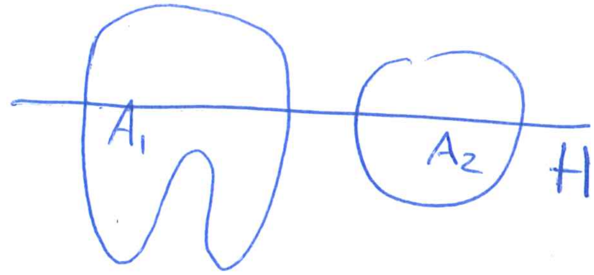
Proof: By approximation can assume g smooth.

If $\nexists$ such x, consider $f : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$, $f(x) := (\tilde{g}(x), 0)$, where $\tilde{g}(x) := g(x) - g(-x)$.

Then, $\ell := \mathbb{R} e_{n+1}$ satisfies $\ell \cap f(S^n) = \emptyset$ ↯ $W_2(f, 0) = 1$. $\boxtimes$

Cor (Ham-Sandwich) $A_1, \dots, A_n \subset \mathbb{R}^n$ bounded open sets

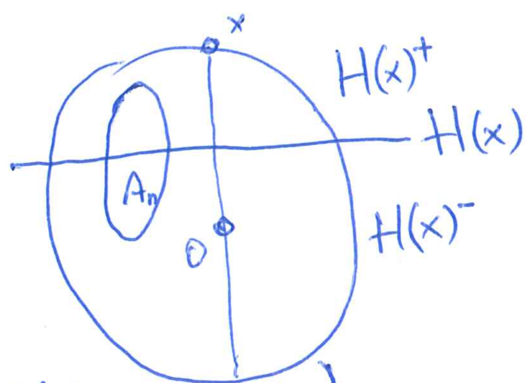
$\Rightarrow \exists$ hyperplane H that simultaneously bisects them.



Proof: Wlog $A_1, \dots, A_n \subset B_1(0)$

For each $x \in S^{n-1}$ let $H(x)$ be the hyperplane orthogonal to $\mathbb{R}x$ that bisects A_n.

Let $H(x)^{\pm} \subset \mathbb{R}^n$ be the halfspace above/below.



Consider $g : S^{n-1} \longrightarrow \mathbb{R}^{n-1}$

$$g(x) := \left(\mathrm{Vol}(A_1 \cap H(x)^+), \dots, \mathrm{Vol}(A_{n-1} \cap H(x)^+) \right).$$

Borsuk-Ulam $\Rightarrow \exists x \in S^{n-1} : g(-x) = g(x) \Rightarrow$ assertion. $\boxtimes$

Cor (Lusternik-Schnirelmann)

$U_1, \dots, U_{n+1} \subset S^n$ open cover $\Rightarrow \exists i : U_i \cap (-U_i) \neq \emptyset$.

Proof: Suppose $U_1, \dots, U_{n+1} \subset S^n$ open cover s.t. $U_i \cap (-U_i) = \emptyset$ $\forall i$.

$\Rightarrow \exists$ closed cover $C_1, \dots, C_{n+1} \subset S^n$ s.t. $C_i \cap (-C_i) = \emptyset$ $\forall i$.

(cf. partition of unity)

Consider $g : S^n \to \mathbb{R}^n$, $g(x) := (d(x, C_1), \dots, d(x, C_n))$.

Borsuk-Ulam $\Rightarrow \exists x \in S^n : g(-x) = g(x)$.

If i-th entry of $g(x)$ is 0, then $x, -x \in C_i$, done.

If no entry of $g(x)$ is 0, then $x, -x \notin \bigcup_{i=1}^{n} C_i \Rightarrow x, -x \in C_{n+1}$.

$\boxed{//}$

Rmk: There are many further applications of the Borsuk-Ulam thm.

e.g. $K(n,k)$ = Kneser graph

$\binom{n}{k}$ vertices corresponding to $A \subseteq \{1, \dots, n\}$ of size k.

Edge between A & B $:\Leftrightarrow A \cap B = \emptyset$.

Thm (Lovasz) $\chi(K(n,k)) = \begin{cases} n - 2k + 2, & n \geq 2k \\ 1, & n < 2k \end{cases}$.

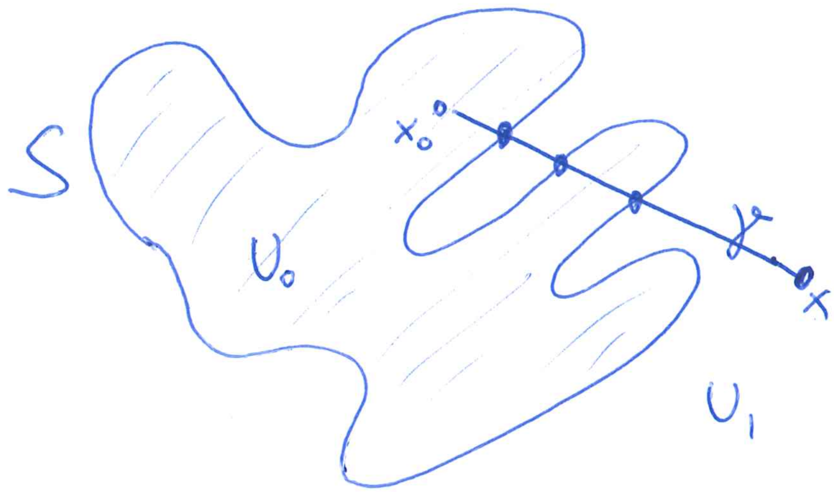
chromatic number $\rightsquigarrow$ "topological combinatorics".

Using similar degree arguments one can prove:

Jordan-Brouwer separation thm

The complement of any compact connected smooth hypersurface $S \subset \mathbb{R}^n$ consists of two connected open sets, the "inside" U_0 & the "outside" U_1.
Moreover, $\bar{U}_0$ is a cpt mfd with bdry $\partial \bar{U}_1 = S$.

Idea:



$$d : \mathbb{R}^n \setminus S \longrightarrow \mathbb{Z}_2$$
$$x \longmapsto \# \gamma^{-1}(S) \bmod 2,$$

where γ is any smooth curve from x_0 to x that is transverse to S.

$$U_i := \{ x \in \mathbb{R}^n \setminus S : d(x) = i \}.$$

Rmk (Alexander horned sphere) $\exists$ cont. embedded S^2 in $\mathbb{R}^3$ s.t. U_1 is not simply connected.
look up on Wikipedia!

Orientations & integer degree

Idea: Count with signs according to orientations to get $\deg(f) \in \mathbb{Z}$.

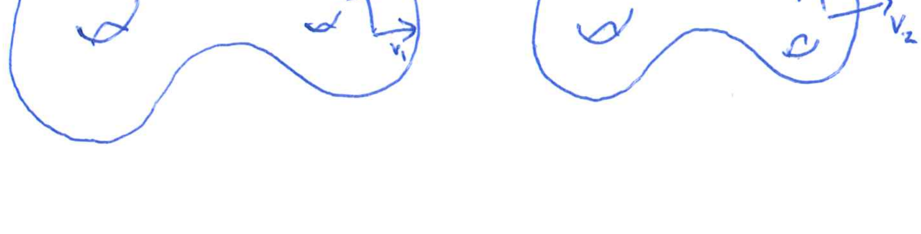
Basic notions:

-) V n-dim real vector space
  basis $(v_1, \dots, v_n) \sim (v_1', \dots, v_n') :\Leftrightarrow v' = Av$ with $\det(A) > 0$.
  orientation of $V := [(v_1, \dots, v_n)]$ ← equivalence class.
  So V has exactly 2 orientations.
  (convention: for $n = 0$ orientations $+1, -1$)

-) orientation of mfd M given by orientations for T_pM, depending continuously on p, i.e. $\exists$ charts $\varphi : U \to \mathbb{R}^n$, s.t. $d\varphi_p : T_pM \to \mathbb{R}^n$ is orientation preserving $\forall p \in U$.

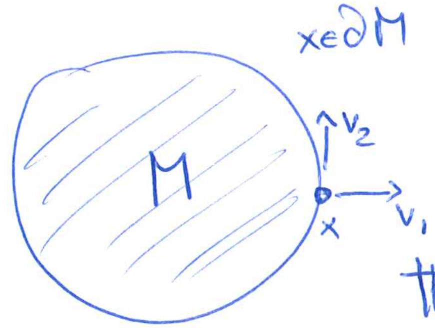
Ex
-) $\mathbb{R}^2, S^2, T^2, T^2 \# T^2, \dots$ are orientable
-) Möbius band, $\mathbb{RP}^2, \mathbb{RP}^2 \# \mathbb{RP}^2, \dots$ are not orientable

Note: Any connected orientable mfd admits exactly 2 orientations.



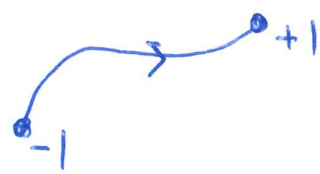
Orientation for $M \rightsquigarrow$ Orientation for ∂M;

$\underline{n \geq 2:}$



$(v_1, \dots, v_n)$ positively oriented basis for $T_x M$, st v_1 is outward. then $(v_2, \dots, v_n)$ pos. oriented basis for $T_x \partial M$.

$\underline{n = 1:}$



__Ex__ $S^2 \subset \mathbb{R}^3$ oriented as bdry of unit ball.

More generally, by Jordan-Brouwer get orientation for any compact hypersurface in $\mathbb{R}^n$.

Now, let $\underset{\text{cpt}}{M^n}, \underset{\text{connected}}{N^n}$ be oriented mfds (without bdry), $f: M \to N$ smooth, $x \in M$ regular point.

$\Rightarrow df_x: T_x M \to T_{f(x)} N$ is a linear isomorphism between oriented vector spaces.

$$\operatorname{sign}(df_x) := \begin{cases} +1 & \text{if } df_x \text{ is orientation preserving} \\ -1 & \text{if } df_x \text{ is orientation reversing.} \end{cases}$$

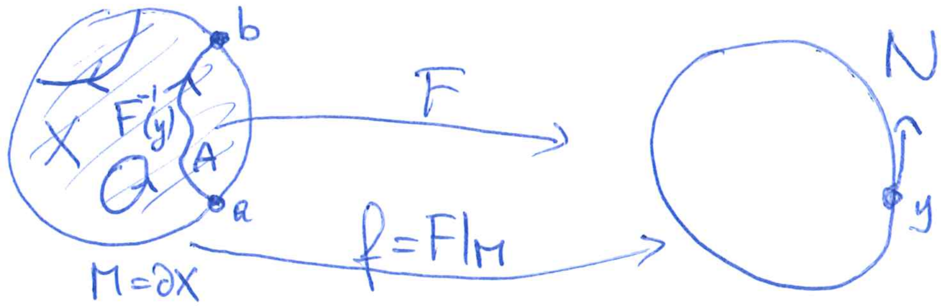
__Thm__ (integer degree) Let M, N, f as above. Then:

- $\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(df_x) \in \mathbb{Z}$ is well-defined, i.e. independent of the choice of regular value $y \in N$.
- If f is smoothly homotopic to g, then $\deg(f) = \deg(g)$.

__Rmk:__ As usual, can generalize to continuous setting by approximation.

<u>Lemma</u> If $f: M \to N$ extends to a smooth map $F: X \to N$, where $\partial X = M$, then $\deg(f;y) = 0$ for any $y \in N$ regular.

<u>Proof of Lemma:</u> Can assume y is regular for F as well.



$F^{-1}(y)$ = finite union of arcs & circles ; $\partial F^{-1}(y) \subset \partial X = M$.

Let A be one of these arcs. Write $\partial A = \{a\} \cup \{b\}$.

Orientations for X & N determine orientation for A:

$x \in A$. $(v_1, \dots, v_{n+1})$ pos. or. basis for $T_x X$.

v_1 determines required orientation for $A \iff$

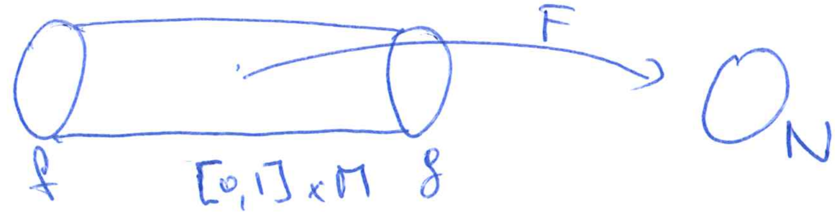
dF_x carries $(v_2, \dots, v_{n+1})$ to pos. or. basis for $T_y N$.

Thus, $\operatorname{sign}(df_b) = +1$, $\operatorname{sign}(df_a) = -1$

sum over arcs $\Rightarrow \deg(f;y) = 0$. $\square$

Proof of thm

-) Suppose $F : [0,1] \times M \to N$ smooth homotopy between f & g.



$$\partial([0,1]\times M) = \underbrace{\{0\}\times M}_{\text{with wrong orientation}} \cup \underbrace{\{1\}\times M}_{\text{with correct orientation}}$$

So $\deg(g;y) - \deg(f;y) = \deg(F|_{\partial([0,1]\times M)}) = 0$

by lemma.

-) Now suppose $y, z \in N$ both regular for $f : M \to N$.

Let $h : N \to N$ diffeo, st $h(y) = z$ & h isotopic to id
(in part. h orient. pres.)

$\Rightarrow \deg(f;y) = \deg(h\circ f; z) = \deg(f;z)$ $\boxed{\checkmark}$

Ex
-) $S^1 \to S^1$, $z \mapsto z^k$ has $\deg(f) = k \in \mathbb{Z}$.
-) $f : M \to N$ diffeo has $\deg(f) = +1/-1$ iff f is orientation preserving/reversing

In part, antipodal map $a : S^n \to S^n$, $x \mapsto -x$

is $a(x) = r_1 \circ \dots \circ r_{n+1}(x)$, where $r_j(x_1, \dots, x_{n+1}) = (x_1, \dots, -x_j, \dots, x_{n+1})$

so $\deg(a) = (-1)^{n+1}$

Hence, $a \not\simeq \mathrm{id}_{S^n}$ for n even! (detected by deg, but not by $\deg_2$).

Thm (hairy ball)

S^n admits a nowhere vanishing vector field $\Leftrightarrow$ n odd.

Proof: If $n = 2k-1$, then $v(x_1, \dots, x_{2k}) = (x_2, -x_1, \dots, x_{2k}, -x_{2k-1})$ defines a nowhere vanishing vf on S^n (indeed $x \cdot v(x) = 0$)

Now n even. Suppose $\exists$ nowhere van vf v.

Can assume $|v| = 1$ (otherwise replace v by $\frac{v}{|v|}$).

Then $F: S^n \times [0,1] \to S^n$

$$F(x,t) = x\cos(\pi t) + v(x)\sin(\pi t)$$

is a smooth homotopy between id_{S^n} & a ↯ $\deg(a) = -1$ ☐

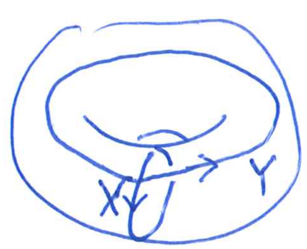
Rmk: Using similar methods can define intersection number

$$I(f, Y) \in \mathbb{Z} \text{ with } I(f, \{y\}) = \deg(f)$$

In particular, for $f = i_X : X \hookrightarrow M$,

this yields $X \cdot Y := I(i_X, Y) \in \mathbb{Z}$

Ex: •)



$X \cdot Y = 1 = -Y \cdot X$

$X . X = 0 = Y \cdot Y$

•) X = zero section $\subset$ Möbius band

has self intersection number $X \cdot X = -1$.