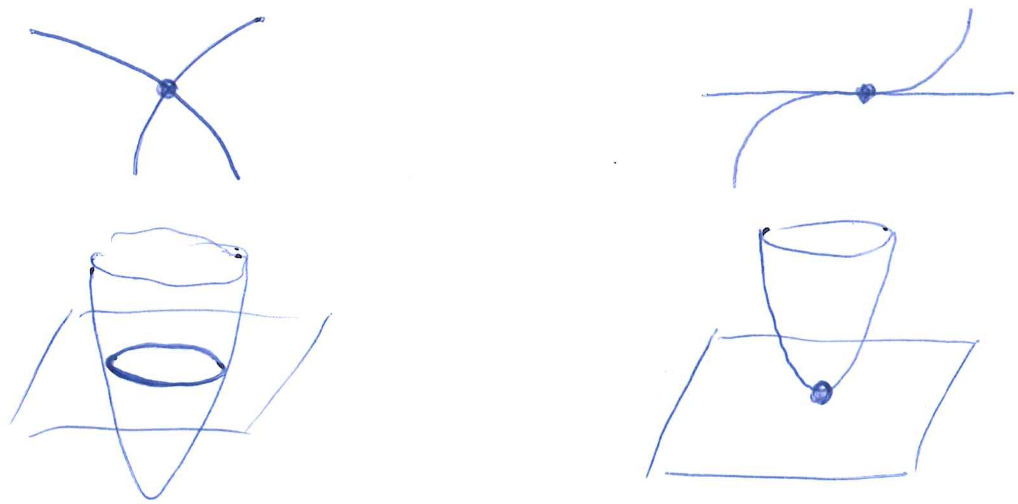
Transversality & genericity

Idea: transversal — not transversal



Def: Two submanifolds $X, Y \subset M$ are transversal if $\forall p \in X \cap Y$: $\quad T_pX + T_pY = T_pM$.

Prop: $X, Y \subset M$ transversal submfds
$\Rightarrow X \cap Y$ is a submfd with $\mathrm{codim}_M(X \cap Y) = \mathrm{codim}_M X + \mathrm{codim}_M Y$.

Here: $\mathrm{codim}_M X = \dim M - \dim X$.

Intuition: $X = \{f_1 = \dots = f_k = 0\}$, $Y = \{g_1 = \dots = g_\ell = 0\}$ $\Rightarrow X \cap Y$ cut out by $k + \ell$ functions.

We'll prove a more general result:

<u>Def</u>: A smooth map $f: X \to M$ is transversal to a submfd $Y \subset M$ if $\forall x \in f^{-1}(Y)$:

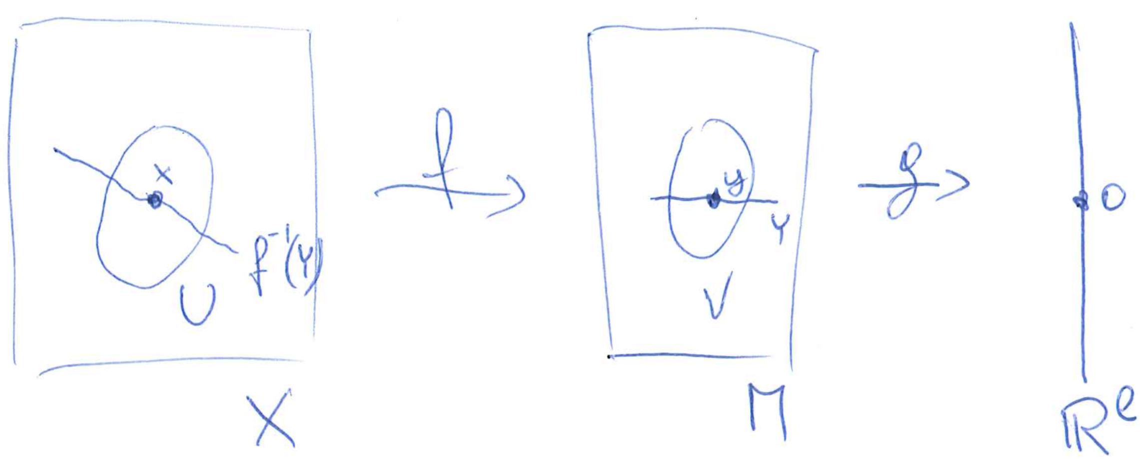
$$\text{Image}(df_x) + T_{f(x)}Y = T_{f(x)}M.$$

<u>Note</u>: i_X transversal to $Y \iff X$ transversal to Y.

<u>Thm</u> If a smooth map $f: X \to M$ is transversal to a submfd $Y \subset M$, then $f^{-1}(Y) \subset X$ is a submfd. Moreover, $\text{codim}_X f^{-1}(Y) = \text{codim}_M Y$.

<u>Note</u>: Have seen this before in case Y = point.

<u>Proof</u>:



For $y \in Y$ $\exists V \ni y$ open nbd & $g : V \to \mathbb{R}^\ell$ smooth, where ℓ = codim of Y in M, s.t. $Y \cap V = g^{-1}(0)$.

If $y = f(x)$, then $\exists U \ni x$ open nbd, s.t.

$$f^{-1}(Y) \cap U = (g \circ f|_U)^{-1}(0).$$

Now, $d(g \circ f)_x = dg_y \circ df_x$ is surjective $\Longleftrightarrow$

$$\mathrm{Image}(df_x) + T_y Y = T_y M.$$

Submersion thm for $g \circ f$ $\Rightarrow$ assertion. $\boxed{/\!/\!/}$

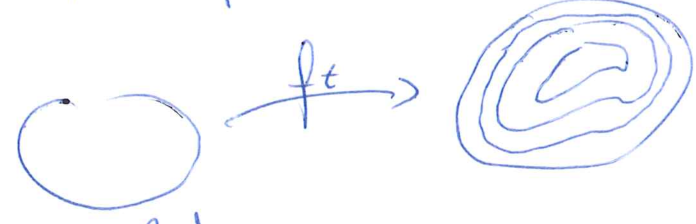
<u>Idea:</u> Transversal maps are generic, i.e. open & dense in suitable space of maps.

"stability" ↑ (open)   "Sard's thm" ↑ (dense)

<u>Def:</u> $f_0 : X \to Y$ and $f_1 : X \to Y$ are <u>smoothly homotopic</u> / <u>homotopic</u> if $\exists F : X \times [0,1] \to Y$ smooth/continuous, s.t.

$$F(x,0) = f_0(x) \text{ \& } F(x,1) = f_1(x).$$

<u>Notation:</u> $f_t(x) \equiv F(x,t)$



<u>Note:</u> That's an equivalence relation.

Ex •) Any $f_0, f_1 : X \to \mathbb{R}^n$ are homotopic.

Indeed, just set $f_t(x) = (1-t) f_0(x) + t f_1(x)$

(similarly, for maps to any contractible space Y)

•) $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^2 \setminus \{0\}$
are homotopic

$\Longleftrightarrow$ they have the same winding number.

•) $f_0, f_1 : \{p\} \to Y$ are homotopic $\Longleftrightarrow$

$f_0(p)$ & $f_1(p)$ are in the same path connected component of Y.

Def: A property of a smooth map $f_0 : X \to Y$ is stable if $\forall$ smooth homotopy $f_t : X \to Y$ $\exists \varepsilon > 0$ st each f_t with $t < \varepsilon$ also posesses this property.

<u>Prop</u> The following classes of smooth maps of a compact mfd X into a mfd Y are stable:

(a) local diffeos

(b) immersions

(c) submersions

(d) maps transversal to any specified closed submfd $Z \subset Y$.

(e) embeddings

(f) diffeos.

<u>Proof</u>: (a) - (d) follows from compactness of X,

since $\{\text{matrices of maximal rank}\} \subset M_{n\times m}(\mathbb{R})$ open.

(e) enough to show injectivity preserved.

If not, $\exists\, t_i \to 0$, $x_i \neq \bar{x}_i$ st $f_{t_i}(x_i) = f_{t_i}(\bar{x}_i)$ $(*)$

X cpt, f_0 injective $\Rightarrow$ $x_i, \bar{x}_i \longrightarrow x_0$

Consider $F(x,t) := (f_t(x), t)$, $F : X \times I \to Y \times I$

$$\Rightarrow dF_{(x_0,0)} = \left(\begin{array}{c|c} d(f_0)_{x_0} & * \\ \hline 0 & 1 \end{array}\right) \text{ invertible}$$

inverse fn thm $\Rightarrow$ $\lightning$ with $(*)$ for i large.

(f) follows from (e). $\square$

<u>recall:</u> $f : M^m \to N^n$ smooth

$C = \{x \in M : df_x \text{ not surjective}\}$ <u>critical points</u>

$f(C) \subset N$ <u>critical values</u>

<u>Thm</u> (Sard) If $f : M \to N$ is smooth, then a.e. $y \in N$ is a regular value, i.e. $f(C) \subset N$ has measure zero.

Here, $A \subset N$ <u>measure zero</u> means
$\forall$ charts (U, φ) the set $\varphi(A \cap U) \subset \mathbb{R}^n$ has measure zero.

<u>recall:</u> $S \subset \mathbb{R}^n$ has measure zero $:\Leftrightarrow$
$\forall \varepsilon > 0 \ \exists B_{r_i}(x_i)$ s.t. $S \subset \bigcup_i B_{r_i}(x_i)$ & $\sum_i r_i^n < \varepsilon$.

<u>Note:</u> S_i measure zero $\Rightarrow \bigcup_i S_i$ measure zero.

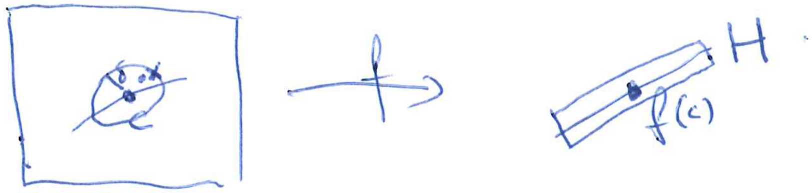
So it suffices to prove Sard's theorem for

$$\mathbb{R}^m \supset [0,1]^m \xrightarrow{f} \mathbb{R}^n.$$

Cases:

1. $m < n$ trivial: $\dim_H f(\mathbb{R}^m) \underset{f \text{ Lipschitz (loc)}}{\le} m < n$.

2. $m = n$ easy:



If $c \in C$, then $\text{image}(df_c) \subset H$

for some hyperplane $H \subset T_{f(c)}\mathbb{R}^n$.

$\forall \varepsilon > 0 \ \exists \delta > 0$:

If $|x - c| \le \delta$, then $|f(x) - f(c)| \le \overset{\text{Lipschitz constant}}{K}\delta$

and $d(f(x), H) \le \varepsilon\delta$ (since f is C^1)

Thus, $\mathrm{Vol}(f(B_\delta(c))) \le (2\varepsilon\delta)(2K\delta)^{n-1}$

ε arbitrary & covering $\Rightarrow$ $f(C)$ has measure zero.

3. $m > n$ hard:

this relies on higher differentiability of f.

Rmk: More precisely, if $f : \mathbb{R}^m \to \mathbb{R}^n$ is C^k,

then $\dim_H f(\{x : \mathrm{rk}(df_x) \le r\}) \le r + \frac{m-r}{k}$.

(in part: $k > m - n + 1 \Rightarrow f(C)$ has measure zero)

Proof of Sard's thm

$$f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

In addition to $C = \{x \in \mathbb{R}^m \mid df_x \text{ is not surjective}\}$

consider $C_k := \{x \in C \mid$ all partial derivatives of f of order $\le k$ vanish at $x \}$.

Note: If $k > \frac{m}{n} - 1$, then $f(C_k)$ has measure zero.

Indeed: for $r \le \delta$: $|x - c| \le r \Rightarrow |f(x) - f(c)| \le K \cdot r^{k+1}$

$$\text{So } \dim_H f(C_k) \le \frac{\dim_H C_k}{k+1} \le \frac{m}{k+1} < n.$$

Hence, suffices to show:

(A) $f(C - C_1)$ has measure zero

& (B) $f(C_\ell - C_{\ell+1})$ has measure zero for $1 \le \ell \le k$.

A) Let $c \in C \setminus C_1$. Can assume $\left.\frac{\partial f_1}{\partial x_1}\right|_c \neq 0$.

Change of variables: $\tilde{x}_1 = f_1(x_1, \dots, x_m)$, $\tilde{x}_2 = x_2, \dots, \tilde{x}_m = x_m$

$\Rightarrow f(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m) \in \{\tilde{x}_1\} \times \mathbb{R}^{n-1}$

get maps $f_t : (\{t\} \times \mathbb{R}^{m-1}) \cap V \longrightarrow \{t\} \times \mathbb{R}^{n-1}$

Note that $\frac{\partial f_i}{\partial \tilde{x}_j} = \left(\begin{array}{c|c} 1 & 0 \\ \hline * & Df_t \end{array} \right)$

Induction on dimension & Fubini $\Rightarrow f(C \setminus C_1)$ has measure zero.

B) $c \in C_\ell \setminus C_{\ell+1}$.

Can assume $w(x) := \frac{\partial^\ell f_1}{\partial x_{i_1} \cdots \partial x_{i_\ell}}$ satisfies $w(c) = 0$, $\left.\frac{\partial w}{\partial x_1}\right|_c \neq 0$.

As before, set $\tilde{x}_1 = w(x_1, \dots, x_m)$, $\tilde{x}_2 = x_2, \dots, \tilde{x}_m = x_m$.

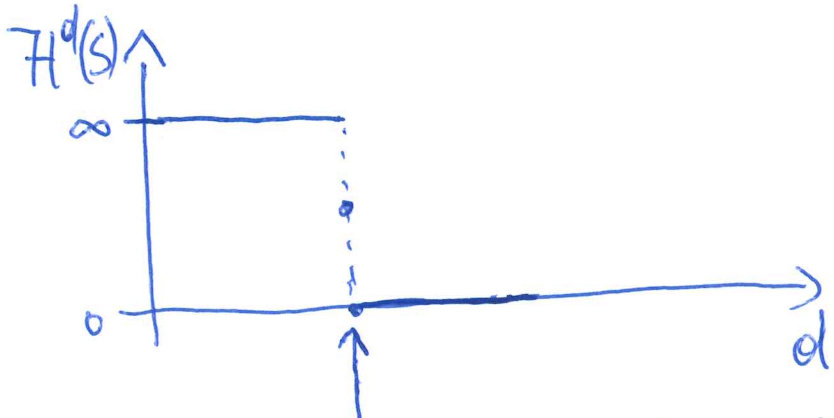
In the new variables we have $C_\ell \cap V \subset \{0\} \times \mathbb{R}^{m-1}$

Induction on dimension $\Rightarrow f(C_\ell \setminus C_{\ell+1})$ has measure zero. $\boxed{\checkmark}$

Reminder: Hausdorff measure & Hausdorff dimension

$S \subset \mathbb{R}^n$ (or metric space)

$$\mathcal{H}^d(S) := \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} r_i^d \;\middle|\; S \subset \bigcup_{i=1}^{\infty} B_{r_i}(x_i),\ r_i \le \delta \right\}$$



$$\dim_H(S) := \inf\{ d \ge 0 \mid \mathcal{H}^d(S) = 0 \}.$$

Note if $|f(x) - f(y)| \le K |x-y|$,

then $\mathcal{H}^d(f(S)) \le K^d \mathcal{H}^d(S)$.

In particular: $\dim_H f(S) \le \dim_H S$.

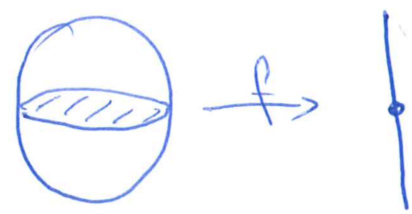
Also note that: $\dim_H S < n \Rightarrow S$ has measure zero.

Applications of Sard's thm

.) Brouwer fixed point thm

.) transversality is generic

First: stuff about boundaries



Prop: $M^m \xrightarrow{f} N^n$ smooth. Set $\partial f := f|_{\partial M} : \partial M \to N$.
(M mfd with bdry ∂M)

Let $Y \subset N$ be a submfd without bdry.

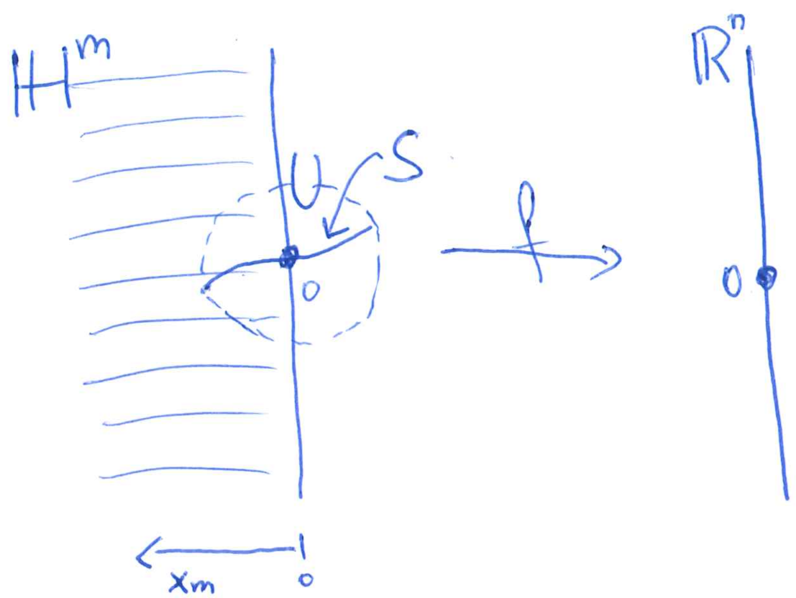
If $f \pitchfork Y$ & $\partial f \pitchfork Y$, then $f^{-1}(Y)$ is a mfd with bdry $\partial f^{-1}(Y) = f^{-1}(Y) \cap \partial M$.

Proof: As usual, enough to show this in case $Y = \{y\}$.

Also, already know $f^{-1}(y) \cap \mathrm{Int}(M)$ is mfd of correct dim.

Consider now $x \in f^{-1}(y) \cap \partial M$.

Working in charts, can assume $x = 0 \in \mathbb{H}^m$, $y = 0 \in \mathbb{R}^n$, $f : \mathbb{H}^m \to \mathbb{R}^n$.



f smooth $\Rightarrow$ $\exists$ extension $\tilde{f} : \underset{\uparrow \text{nbd of } 0 \text{ in } \mathbb{R}^m}{U} \to \mathbb{R}^n$.

0 regular value for f, decrease U $\Rightarrow$ 0 reg. value for $\tilde{f}$.

$\Rightarrow$ $S := \tilde{f}^{-1}(0) \subset U$ is $(m-n)$-dim submfd (without bdry).

To show: $S \cap \mathbb{H}$ is mfd with bdry.

by assumption:

$df_0 : \mathbb{R}^m \to \mathbb{R}^n$ & $d(\partial f)_0 : \mathbb{R}^{m-1} \to \mathbb{R}^n$ surjective

$\Rightarrow$ $\ker df_0 \not\subset \mathbb{R}^{m-1}$,

i.e. $\underbrace{\ker df_0}_{= T_0 S} + \underbrace{\mathbb{R}^{m-1}}_{= T_0 \partial\mathbb{H}} = \mathbb{R}^m$

Here $\mathbb{R}^{m-1}$ ~~$\equiv \mathbb{R}^{m-1}$~~ $\equiv \mathbb{R}^{m-1} \times \{0\}$

$d(\partial f)_0 = df_0|_{\mathbb{R}^{m-1}}$

i.e. S & $\partial\mathbb{H}$ intersect transversally at 0.

$\Rightarrow$ assertion. $\boxed{\text{?c}}$

<u>Note:</u> $y \in N$ regular value for $f : M \to N$ & $\partial f : \partial M \to N$

$\Leftrightarrow$ regular value for $f|_{\text{Int}(M)} : \text{Int}(M) \to N$ & $\partial f : \partial M \to N$

since for $x \in \partial M$ have $d(\partial f)_x = df_x|_{T_x(\partial M)}$

Hence, Sard's thm holds for mfds with bdry as well.

Let $D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$

<u>Thm</u> (Brouwer) Every continuous $f : D^n \to D^n$ has a fixed point, i.e. $\exists x \in D^n$ st $f(x) = x$.

<u>Lemma</u> M cpt smooth mfd with bdry $\partial M \neq \emptyset \Rightarrow \nexists$ no smooth retraction from M onto its bdry, i.e. $\nexists f : M \to \partial M$ smooth with $f|_{\partial M} = id_{\partial M}$.

<u>Proof of Lemma:</u> Suppose $\exists$ such f.

Sard $\Rightarrow \exists y \in \partial M$ a regular value for f (hence also regular for ∂f)

Prop $\Rightarrow f^{-1}(y)$ is a 1-dim cpt mfd with

$$\partial f^{-1}(y) = f^{-1}(y) \cap \partial M = \{y\}$$

But $f^{-1}(y) =$ disjoint union of $[0,1]$'s & S^1's

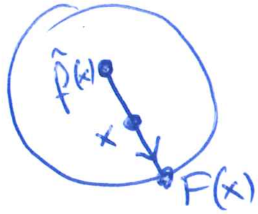
$\Rightarrow \# \partial f^{-1}(y) =$ even ↯ $\boxed{}$

<u>Proof of thm</u> Suppose $\exists f : D^n \to D^n$ cont. without f.p.

D^n cpt $\Rightarrow \inf_{D^n} |f(x) - x| =: \varepsilon > 0$

$\Rightarrow \exists \tilde{f} : D^n \to D^n$ smooth without f.p.

But then $F : D \to \partial D$ defined as



is smooth with $F|_{\partial D} = id$ ↯ $\boxed{}$

Transversality thm

Suppose $F : M \times S \to N$ smooth
$\qquad\qquad\qquad\qquad\;\; \cup$
$\qquad\qquad\qquad\qquad\;\; Y$ submfd , where only M has bdry.

If $F \pitchfork Y$ & $\partial F \pitchfork Y$, then for a.e. $s \in S$: $f_s \pitchfork Y$ & $\partial f_s \pitchfork Y$.

Proof: $W := F^{-1}(Y) \subset M \times S$ is submfd with bdry $\partial W = W \cap (\partial M \times S)$.

By Sard's thm, enough to show for $s \in S$ we have:

(i) s regular value for $\pi : W \to S \Rightarrow f_s \pitchfork Y$.

(ii) s regular value for $\partial\pi : \partial W \to S \Rightarrow \partial f_s \pitchfork Y$.

So suppose $\underbrace{f_s(x)}_{\equiv F(x,s)} = y \in Y$.

$F \pitchfork Y \Rightarrow \text{Image}(dF_{(x,s)}) + T_y Y = T_y N$,

i.e. $\forall a \in T_y N \;\; \exists b = (w, e) \in T_{(x,s)}(M \times S)$:

$$dF_{(x,s)}(b) - a \in T_y Y.$$

Given $a \in T_y N$ need to find $v \in T_x M$ st. $d(f_s)_x(v) - a \in T_y Y$.

If $e = 0$, done since $dF_{(x,s)}(w, 0) = d(f_s)_x(w)$.

If $e \neq 0$, then can still argue as follows:

$$\begin{tikzcd} T_xM \times T_sS \arrow[r] & T_sS \ni e \\ T_{(x,s)}W \arrow[u, hook] \arrow[ur, "d\pi_{(x,s)}"'] & \end{tikzcd}$$

$d\pi_{(x,s)}$ surjective by assumption.

$\Rightarrow \exists c \in T_{(x,s)}W : d\pi_{(x,s)}(c) = e.$

And c must have form $c = (u, e)$ for some $u \in T_xM$

Then $v := w - u \in T_xM$ satisfies

$$d(f_s)_x(v) - a = dF_{(x,s)}[(w,e)-(u,e)] - a$$

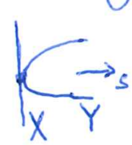
$$= \underbrace{dF_{(x,s)}(b) - a}_{\in T_yY} - \underbrace{dF_{(x,s)}(c)}_{\in T_yY} \in T_yY \Rightarrow (i).$$

Finally, (ii) follows from (i) applied to $\partial F : \partial M \times S \to N.$ $\boxed{///}$

<u>Cor:</u> $f : M \to N$ smooth, $Y \subset N$ cpt submfd, where only M has bdry (possibly $\emptyset$).

$\Rightarrow \exists g : M \to N$ arbitrarily close to f & smoothly homotopic to f,

st. $g \pitchfork Y$ & $\partial g \pitchfork Y$.



<u>Proof:</u> If $N = \mathbb{R}^n$ simply apply thm to $F : M \times \mathbb{R}^n \to \mathbb{R}^n$

$F(x,s) = f(x) + s.$

In general, $N \subset \mathbb{R}^k$ by Whitney.

Apply thm to $F : M \times B^k_\varepsilon \to \mathbb{R}^k$, $(x,s) \mapsto \pi(f(x)+s)$ where $\pi : N^\varepsilon \to N$ is the nearest neighbor projection. $\boxed{}$