Embeddings, immersions, submersions

<u>Idea:</u> Smooth maps are approximated at small scales by linear maps.

For linear map L, choosing suitable bases we have

(i) bijective $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n)$

(ii) injective $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$

(iii) surjective $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k)$

<u>Def:</u> A smooth map $F: M \to N$ is called a

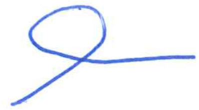
(i) <u>local diffeomorphism</u> if dF_p is bijective $\forall p \in M$

(ii) <u>immersion</u> if dF_p is injective $\forall p \in M$

(iii) <u>submersion</u> if dF_p is surjective $\forall p \in M$.

<u>Ex</u> (i) $\mathbb{R} \to S^1$, $t \mapsto e^{it}$ is a local diffeo (but not bijective)

(ii) $\gamma: I \to M$ smooth curve is immersion $\Leftrightarrow$ $\dot{\gamma}(t) \neq 0$ $\forall t \in I$

e.g.  but not 

(iii) $TM \xrightarrow{\pi} M$ & Hopf map: $S^3 \to S^2$ are submersions.

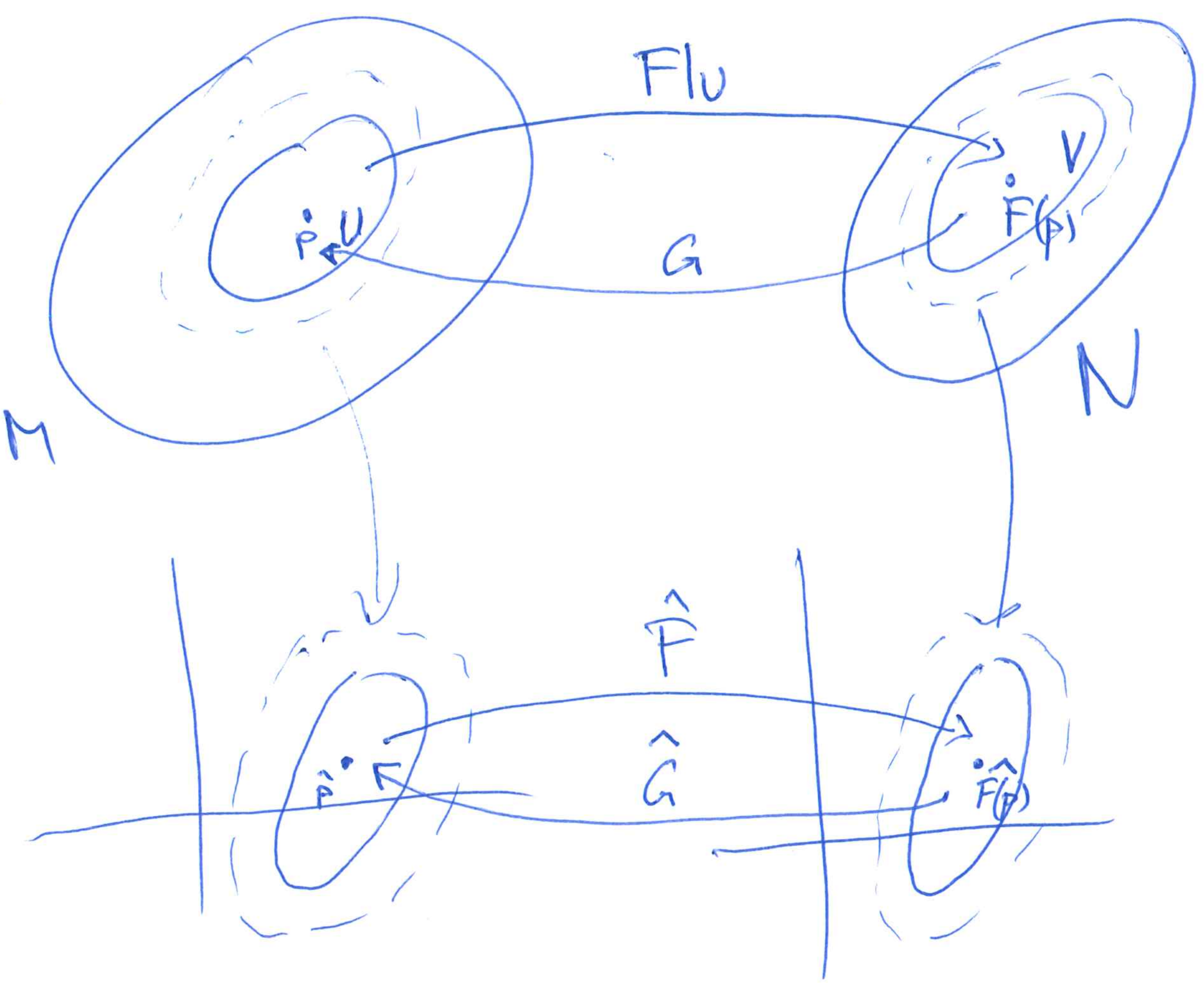
Inverse function thm

Suppose $F: M \to N$ is smooth & dF_p is bijective.

Then F is smoothly invertible near p, i.e.

$\exists\, U \ni p$ open nbd with $V := F(U)$ open
& $G : V \to U$ smooth, st. $G \circ F|_U = id_U$, $F \circ G = id_V$.

Proof:



Using charts, this follows from the inverse fn thm in $\mathbb{R}^n$. ∎

<u>Cor</u> $F: M \to N$ smooth, dF_p bijective

$\Rightarrow \exists$ coords near p & $F(p)$ st. $\hat{F}(x_1,\dots,x_n) = (x_1,\dots,x_n)$.

<u>Q:</u> When can we upgrade local diffeo to global diffeo?

<u>Cor:</u> $F: M \to N$ smooth, bijective & dF_p bijective $\forall p \in M$

$\Rightarrow F$ diffeo.

<u>Proof:</u> F bijective $\Rightarrow \exists F^{-1}: N \to M$.

Near $q = F(p) \in N$ ~~by the functio~~

the map F^{-1} agrees with $(F|_U)^{-1} = G$

from Inverse fn. thm, which is smooth $\boxtimes$

<u>Cor</u> $\underset{\text{compact}}{F: M} \to \underset{\text{connected}}{N}$ smooth, injective & dF_p bijective $\forall p$

$\Rightarrow F$ diffeo.

<u>Proof</u> $K \subset M$ cpt $\Rightarrow F(K) \subset N$ cpt.

$\Rightarrow F(M) \subseteq N$ is open & closed

N connected $\Rightarrow F(M) = N$, i.e. F surjective $\underset{\text{prev. Cor.}}{\Longrightarrow}$ Claim. $\boxtimes$

Local immersion thm

Suppose $f: M \to N$ is smooth & df_p is injective.
Then $\exists$ coords near p & $f(p)$ st. $\hat{f}(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0)$.

Proof: Working in local charts, can assume

$$\mathbb{R}^k \supset U \xrightarrow{f} V \subset \mathbb{R}^n, \quad f(0) = 0, \quad df_0 = \left(\frac{I_k}{0}\right).$$

Consider $F : U \times \mathbb{R}^{n-k} \to \mathbb{R}^n$

$$(x, z) \longmapsto f(x) + (0, z)$$

Note that $dF_0 = \left(\begin{array}{c|c} I_k & 0 \\ \hline 0 & I_{n-k} \end{array}\right) = I_n$.

Inverse function thm $\Rightarrow \exists G$ smooth local inverse of F.

$$\Rightarrow \quad (x, z) = G(F(x,z)) = G(f(x) + (0,z))$$

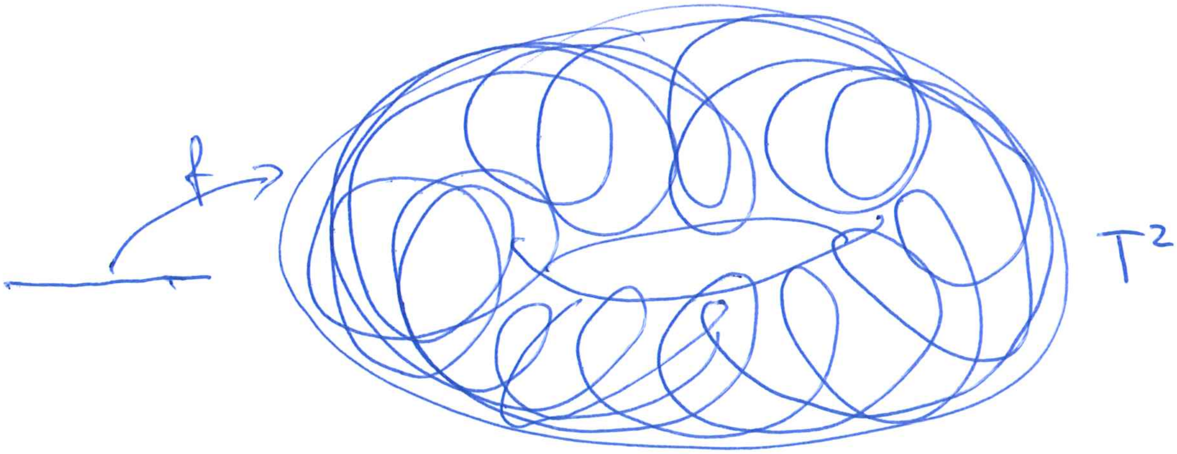
In particular, $\underbrace{G(f(x))}_{\hat{f}(x)} = (x, 0)$

$\square$

Ex $f : \mathbb{R} \longrightarrow T^2 = \mathbb{R}^2/\mathbb{Z}^2$

$x \longmapsto [x, \theta x]$

where $\theta \in (0,1)$ is irrational



f is an injective immersion, but:

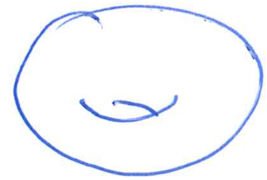
Image $f(\mathbb{R}) \subset T^2$ is dense,

in particular $f(\mathbb{R}) \not\cong \mathbb{R}$.

not homeomorphic

Def: An embedding is an injective immersion, which is a homeomorphism onto its image.

Ex •) $S^n \hookrightarrow \mathbb{R}^{n+1}$


$\subset \mathbb{R}^3$

•) $T^2 \to \mathbb{R}^3$

$[u,v] \longmapsto \big((2+\cos 2\pi u)\cos 2\pi v,\ (2+\cos 2\pi u)\sin 2\pi v,\ \sin 2\pi u\big)$

Q: When is an injective immersion an embedding?

Ex M compact (since then $K \subset M$ closed $\Rightarrow$ $\underset{\text{closed}}{F(K) \subset N}$.)

More generally:

Prop $F: M \to N$ proper injective immersion

$\Rightarrow$ F embedding.

Def: A continuous map $F: X \to Y$ is proper if $F^{-1}(K)$ is cpt $\forall K \subset Y$ cpt.

Idea: "maps ∞ to ∞"

Proof of prop:

$F : M \to N$ proper injective immersion.

enough to show: F is open.

Suppose towards a contradiction $\exists U \subset M$ open,
st $F(U)$ is not open

$\Rightarrow \exists\, y_1, y_2, \ldots \in F(M) \setminus F(U)$ st $y_i \to y \in F(U)$.

F proper $\Rightarrow$ after passing to a subsequence,
we have $x_i := F^{-1}(y_i) \to x \in X$

F injective $\Rightarrow$ $x = F^{-1}(y) \in U$

U open $\Rightarrow$ $x_i = F^{-1}(y_i) \in U$ for large i

$\lightning$

$\square$

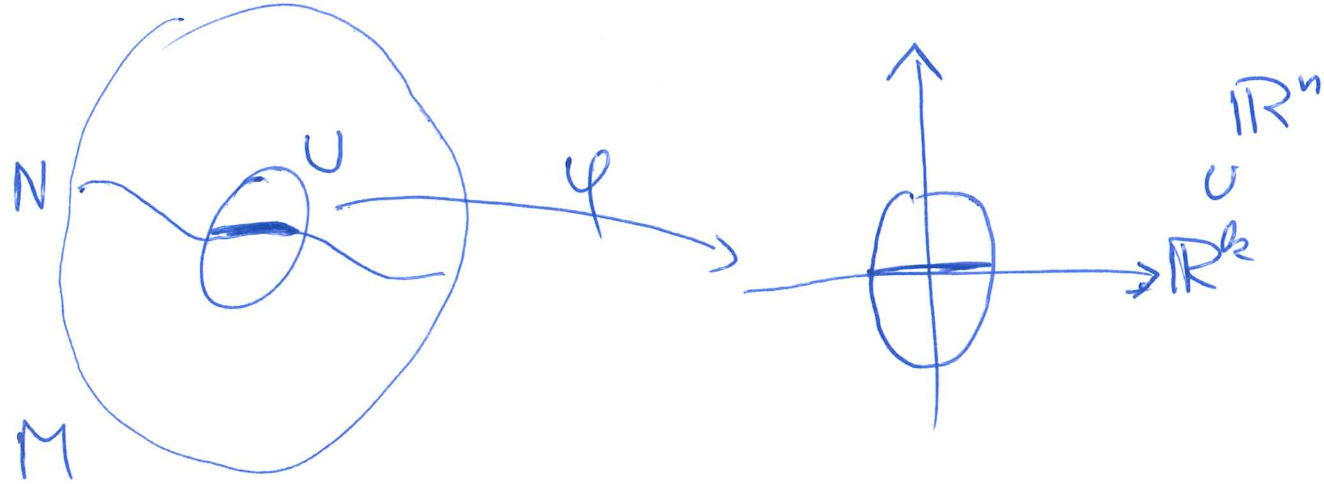
~~Cor $F : M \to N$ embe~~

Def: M^n smooth n-dim mfd

$N \subset M$ is called a k-dim <u>submanifold</u>,

if $\exists$ smooth charts (U, φ) for M,

such that $\varphi(N \cap U) = \mathbb{R}^k \cap \varphi(U)$.



Cor $N^k \xrightarrow{F} M^n$ embedding (e.g. proper injective immersion)

$\Rightarrow F(N) \subset M$ is a k-dim submanifold.

Proof Local immersion thm gives charts as required. $\boxtimes$

Local submersion thm

Suppose $f : M \to N$ is smooth & df_p is surjective.
Then $\exists$ coords near p & $f(p)$ st $\hat{f}(x_1, \dots, x_n) = (x_1, \dots, x_k)$.

Proof Can assume

$$\mathbb{R}^n \supset U \xrightarrow{f} V \subset \mathbb{R}^k, \ f(0) = 0, \ df_0 = (I_k \,|\, 0).$$

Consider $F(x) := (f(x), x_{k+1}, \dots, x_n)$
& apply inverse function thm. $\boxed{/\!/}$

Def: Let $f : M \to N$ be a smooth map.
$c \in N$ is called a regular value of f
if df_p is surjective $\forall p \in f^{-1}(c)$.

Ex: $f : \mathbb{R}^{n+1} \to \mathbb{R}$

$$(x_1, \dots, x_{n+1}) \mapsto x_1^2 + \dots + x_{n+1}^2$$

all $c > 0$ are regular values.

Cor c regular value for $f: M^m \to N^n$, $f^{-1}(c) \neq \emptyset$

$\Rightarrow f^{-1}(c) \subset M$ is a $(m-n)$-dim submanifold.

Moreover, $T_p f^{-1}(c) = \ker(df_p)$ $\forall p \in f^{-1}(c)$.

Proof: Follows from local submersion thm. $\boxtimes$

Ex $U(n) = \{A \in M_n(\mathbb{C}) : A^\dagger A = I\}$

$$f: M_n(\mathbb{C}) \longrightarrow H(n) = \{M : M^\dagger = M\}$$
$$A \longmapsto A^\dagger A$$

$$df_A(B) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (A+\varepsilon B)^\dagger (A+\varepsilon B) = A^\dagger B + B^\dagger A$$

Given $C \in H(n)$, choose $B = \frac{1}{2}AC$. Then $df_A(B) = C$.

Cor $\Rightarrow U(n) = f^{-1}(I) \subset M_n(\mathbb{C})$ is a submfd

of dimension $2n^2 - n^2 = n^2$.

Moreover, $T_I U(n) = \{B : B + B^\dagger = 0\}$,

i.e. Lie algebra = skew Hermitian matrices.

Partitions of unity & Whitney embedding

Idea: Write constant function 1 as sum of bump functions.

Thm (partition of unity)

Let $\{W_\alpha\}$ be an open cover of a smooth mfd M.

Then $\exists\, \eta_i : M \to [0,1]$ smooth, st:

(i) every $x \in M$ has a nbd in which only finitely many η_i's don't vanish.

(ii) $\sum_i \eta_i = 1$

(iii) $\forall i \,\exists \alpha : \overline{\{\eta_i \neq 0\}} \subset W_\alpha$.

Lemma: For any open covering $\{W_\alpha\}$ of M, there exists a locally finite refinement (U_i, φ_i) by good charts, i.e.

(i) every $x \in M$ has a nbd that only intersects finitely many U_i's.

(ii) $\forall i \,\exists \alpha : U_i \subset W_\alpha$

(iii) $\varphi_i(U_i) = B_3(0)$ & $\varphi_i^{-1}(B_1(0))$ still cover.

Lemma $\Rightarrow$ Thm:

Fix $\psi : B_3^n(0) \to [0,1]$ smooth,
st $\psi \equiv 1$ on $\overline{B}_1$ & $\psi \equiv 0$ outside B_2 (cf week 2)

Given $\{W_\alpha\}$, lemma provides (U_i, φ_i).

Set $\psi_i(x) := \begin{cases} \psi(\varphi_i(x)) & x \in U_i \\ 0 & \text{else} \end{cases}$.

Then $\eta_i := \dfrac{\psi_i}{\sum_j \psi_j}$ has the desired properties. $\boxtimes$

Proof of Lemma:

Mfd axioms $\Rightarrow$ can write $M = \bigcup_i K_i$,
where $K_1 \subset K_2 \subset \ldots$ cpt st. $\exists V_i$ open with $K_i \subset V_i \subset K_{i+1}$

If $p \in W_\alpha$, then $p \in K_{i+1} \setminus K_i^\circ$ for some i. $A_i = K_{i+1} \setminus K_i^\circ$

Choose $(U_{p,\alpha}, \varphi_{p,\alpha})$ chart with $U_{p,\alpha} \subset K_{i+2} \setminus K_{i-1}^\circ$ & $\varphi_{p,\alpha}(U_{p,\alpha}) = B_3(0)$ & $U_{p,\alpha} \subset W_\alpha$

Then, the open sets $V_{p,\alpha} := \varphi_{p,\alpha}^{-1}(B_1(0))$ cover the cpt set $K_{i+1} \setminus K_i^\circ$ without leaving $K_{i+2} \setminus K_{i-1}^\circ$.

Choose finite subcover $V_{i,k}$ for each i.

Then $(U_{i,k}, \varphi_{i,k})$ does the job. $\boxtimes$

Prop M cpt smooth mfd

$\Rightarrow \exists$ embedding $M \hookrightarrow \mathbb{R}^N$ for some $N < \infty$.

Proof: M cpt $\Rightarrow \exists\, (U_i, \varphi_i)_{i=1,\dots,k}$ good charts, $\sum_{i=1}^{k} \eta_i = 1$

Define $\Phi : M^n \longrightarrow \mathbb{R}^{k(n+1)}$

$$x \longmapsto (\eta_1(x)\varphi_1(x), \dots, \eta_k(x)\varphi_k(x), \eta_1(x), \dots, \eta_k(x))$$

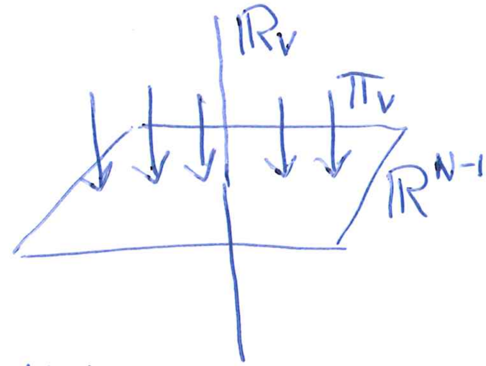
- If $\Phi(x) = \Phi(x')$, choose i st. $x \in \varphi_i^{-1}(B_1(0))$,
  then $\eta_i(x) = \eta_i(x') \neq 0$, hence $\varphi_i(x) = \varphi_i(x')$, so $x = x'$.

- Given $x \in M$, chose i st $x \in \varphi_i^{-1}(B_1(0))$,
  then $d(\underbrace{\eta_i}_{=c}\varphi_i) = c\, d\varphi_i$ injective $\Rightarrow d\Phi_x$ injective

Hence, Φ is an injective immersion.

M cpt $\Rightarrow \Phi$ embedding. $\square$

Consider projection $\pi_v : \mathbb{R}^N \to \mathbb{R}^{N-1} = (\mathbb{R}v)^\perp$



Lemma (dimension reduction)

$M^n \subset \mathbb{R}^N$ embedded submfd.

If $N > 2n+1$, then for a.e. $[v] \in \mathbb{RP}^{N-1}$ the maps

$$\Phi_v : M^n \hookrightarrow \mathbb{R}^N \xrightarrow{\pi_v} \mathbb{R}^{N-1}$$

is an injective immersion.

Proof Let $\Delta_M = \{(p,p) \in M \times M\}$ diagonal

$M_0 = \{(p,0) \in TM\}$ zero-section.

Consider:

$$f : M \times M \setminus \Delta_M \longrightarrow \mathbb{RP}^{N-1}$$
$$(p,q) \longmapsto [p-q]$$

$$g : TM \setminus M_0 \longrightarrow \mathbb{RP}^{N-1}$$
$$(p,w) \longmapsto [w]$$

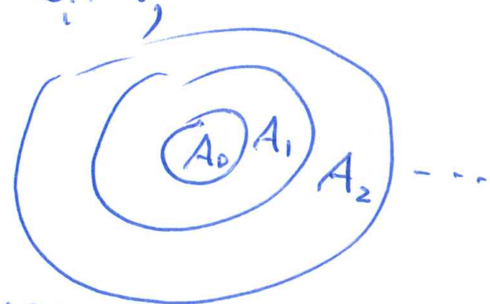
Note that Φ_v injective immersion $\Leftrightarrow$ $[v]$ is not in the image of f & g

Since $N-1 > 2n$ & since (loc.) Lipschitz maps don't increase the (Hausdorff) dimension, this proves the Lemma. $\boxed{\text{//}}$

Thm (Whitney) Any smooth mfd M^n can be embedded in $\mathbb{R}^{2n+1}$.

Proof: For M cpt directly follows by iterating dimension reduction lemma.

For M noncpt, as before decompose $M = \bigcup_i A_i$, where $A_i = \{ i \le \rho \le i+1 \}$ for some smooth proper $\rho : M \to [0, \infty)$



By argument in cpt case $\Rightarrow$ $\exists\, V_i \supset A_i$ open & $\varphi_i : V_i \hookrightarrow \mathbb{R}^{2n+1}$ embedding.

Can assume $V_i \cap V_j = \emptyset$ unless $|i-j| \le 1$.

Choose $\lambda_i : M \to \mathbb{R}$ smooth bump fn, st.

$$\lambda_i \equiv 1 \text{ in nbd of } A_i \ \& \ \lambda_i \equiv 0 \text{ in nbd of } M \setminus V_i .$$

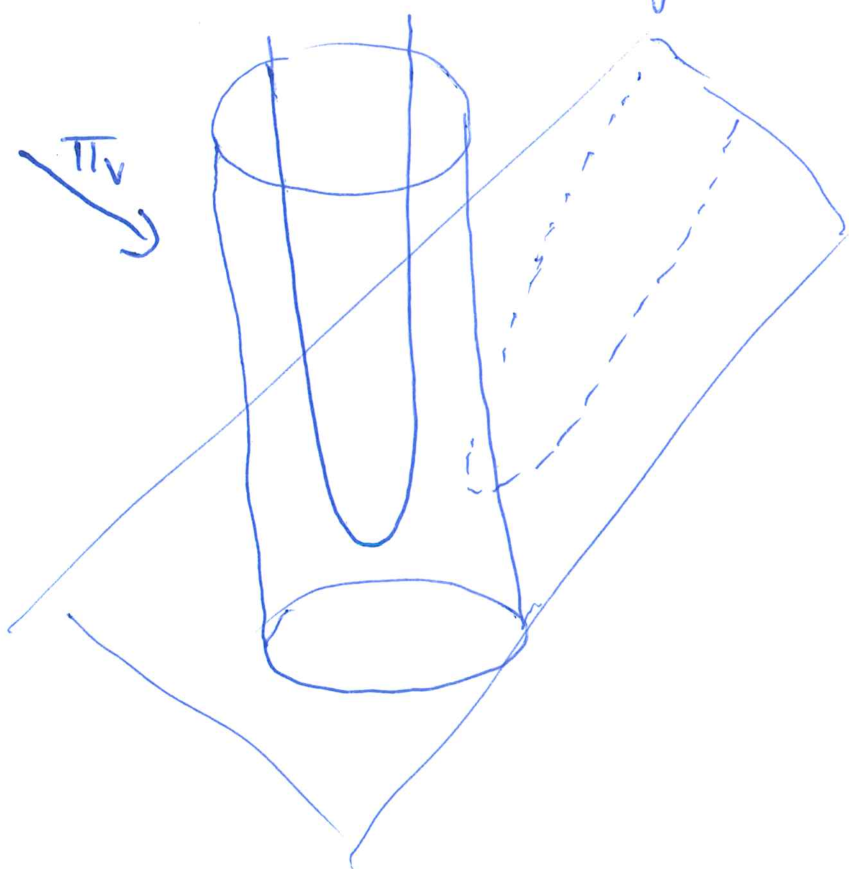
Define $F : M \to \mathbb{R}^{2(2n+1)} \times \mathbb{R}$

$$x \longmapsto \Big(\sum_{i \text{ even}} \lambda_i(x)\varphi_i(x), \sum_{i \text{ odd}} \lambda_i(x)\varphi_i(x), \rho(x) \Big).$$

Then F is a proper injective immersion, hence embedding.

Can arrange that $F(M) \subset \underbrace{B^{2(2n+1)} \times \mathbb{R}}_{\text{"tube"}}$

By elementary geometric argument
π_v with $v \nparallel$ tube preserves properness :



iterate dimension reduction lemma $\Rightarrow$ thm. $\square$

<u>Rmk</u> With more effort ("Whitney trick") can embedd in $\mathbb{R}^{2n}$.