Tangent bundle & vector fields

recall: M smooth mfd. For every $p \in M$ have tangent space T_pM.

~~Idea~~ Goal: put T_pM for all p together into some nice larger space, called the tangent bundle TM

As a set: $TM := \coprod_p T_pM = \{(p,v) \mid p \in M, v \in T_pM\}$

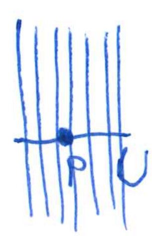
Ex: $T\mathbb{R}^n = \coprod_{p \in \mathbb{R}^n} T_p\mathbb{R}^n \cancel{\cong \coprod_{p\in\mathbb{R}^n} \mathbb{R}^n}$

$$= \{(p,v) \mid p \in \mathbb{R}^n, v \in \underbrace{T_p\mathbb{R}^n}_{\cong \mathbb{R}^n}\} \cong \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$$

In general, want to give TM structure of $2n$-dim smooth mfd.

Projection map

$$\pi : TM \longrightarrow M$$
$$(p,v) \longmapsto p$$

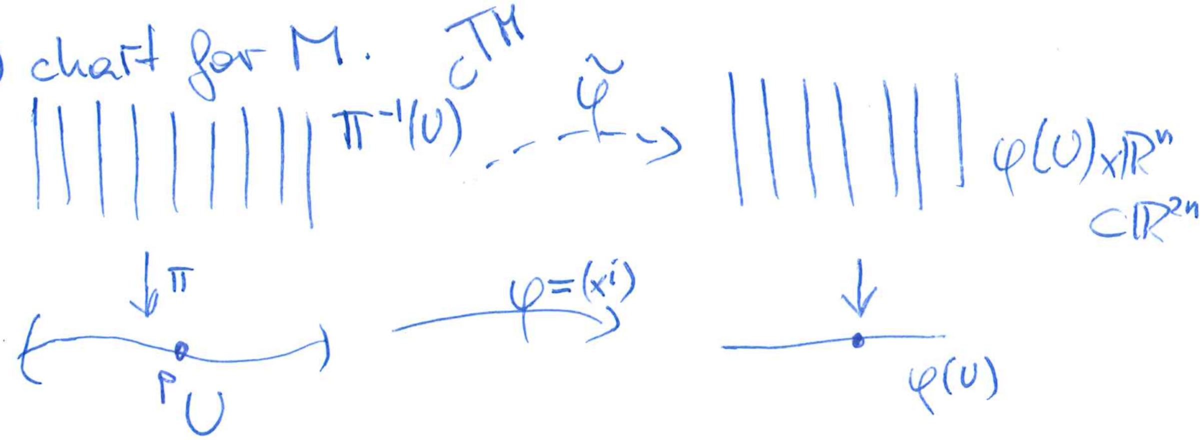


Idea: $TU \cong U \times \mathbb{R}^n$ for small nbd U, but globally can be more complicated!

Prop For any smooth n-dim mfd M, the tangent bundle TM has a natural smooth structure that makes it a $2n$-dim smooth mfd.

Moreover, $\pi : TM \to M$ is smooth.

Proof: (U, φ) chart for M.



$$\pi^{-1}(U) = \{ (p, v) \in TM \mid p \in U \}$$

Define $\tilde{\varphi} : \pi^{-1}(U) \longrightarrow \mathbb{R}^{2n}$ by

$$\tilde{\varphi}\left(\left(p, v^i \frac{\partial}{\partial x^i}\Big|_p\right)\right) := \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n\right)$$

Note that $\tilde{\varphi}$ is bijective onto $\varphi(U) \times \mathbb{R}^n \underset{\text{open}}{\subset} \mathbb{R}^{2n}$,

with $\tilde{\varphi}^{-1}(\underbrace{x^1, \dots, x^n}_{=x}, v^1, \dots, v^n) = \left(\varphi^{-1}(x), \; v^i \frac{\partial}{\partial x^i}\Big|_{\varphi^{-1}(x)} \right).$

Transition maps? Given $(U,\varphi), (V,\psi)$ we have

$$\tilde{\psi}\circ\tilde{\varphi}^{-1}(x^1,\dots,x^n,v^1,\dots,v^n)$$

$$=\left(\tilde{x}^1(x),\dots,\tilde{x}^n(x),\frac{\partial\tilde{x}^1}{\partial x^j}(x)v^j,\dots,\frac{\partial\tilde{x}^n}{\partial x^j}(x)v^j\right)$$

indeed smooth ✓.

M smooth mfd $\Rightarrow \exists\,(U_i,\varphi_i)$ countably many smooth charts, s.t. $\varphi_i^{-1}(B_r^n(q))$ is basis for topology of M.

Define topology on TM by declaring that

$\tilde{\varphi}_i^{-1}(B_r^{2n}(q))$ is basis.

$\Rightarrow$ TM is smooth 2n-dim mfd.
(Hausdorff ✓)

Finally, $\pi: TM \to M$ in coords is just

$(x,v)\mapsto x$, hence smooth. $\boxtimes$

Can put together $dF_p : T_pM \to T_{F(p)}M$ as well:

<u>Prop:</u> If $F : M \to N$ is smooth, then

$$\begin{aligned} dF : TM &\longrightarrow TN \\ (p,v) &\longmapsto (F(p), dF_p(v)) \end{aligned}$$ is also smooth.

<u>Proof:</u> In local coords

$$dF(x^1, \dots, x^n, v^1, \dots, v^n) = \left(F^1(x), \dots, F^m(x), \frac{\partial F^1}{\partial x^i}(x)v^i, \dots, \frac{\partial F^m}{\partial x^i}(x)v^i\right)$$

indeed smooth. $\boxed{\checkmark}$

So $M \xrightarrow{F} N \rightsquigarrow TM \xrightarrow{dF} TN$

Note also:
$$\begin{array}{ccc} TM & \xrightarrow{dF} & TN \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow{F} & N \end{array}$$ commutes.

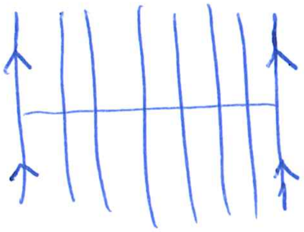
<u>Notation:</u> $(p, v) \equiv v \in TM$.

Q: How does TS^n look like?

recall: $S^1 = [0,2\pi]/0\sim 2\pi$

We have seen that we can glue $[0,2\pi]\times\mathbb{R}$ in two ways:

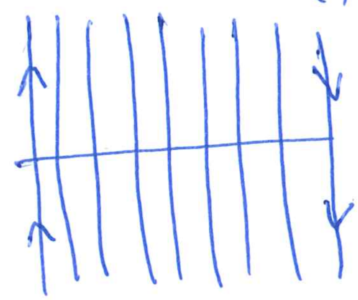
$S^1\times\mathbb{R} = [0,2\pi]\times\mathbb{R}/(0,v)\sim(2\pi,v)$



cylinder

Mobius band

$= [0,2\pi]\times\mathbb{R}/(0,v)\sim(2\pi,-v)$

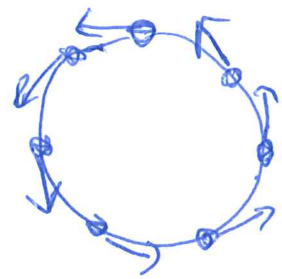


Q: $TS^1 \cong$ cylinder or Mobius band?

$\cong$ diffeomorphic

A: $TS^1 \cong S^1\times\mathbb{R}$

$\left(e^{i\theta}, a\frac{\partial}{\partial\theta}\right) \mapsto \left(e^{i\theta}, a\right)$



$S^1\subset\mathbb{C}$ (complex numbers)

Homework:

$$TS^3 \cong S^3 \times \mathbb{R}^3 \qquad \text{(quaternions)}$$

$$TS^7 \cong S^7 \times \mathbb{R}^7 \qquad \text{(octonions)}$$

Fact (Bott-Milnor, Kervaire)

S^1, S^3, S^7 are the <u>only</u> spheres that are parallelizable, i.e. s.t. $TS^n \cong S^n \times \mathbb{R}^n$ (isomorphic as vector bundle)

<u>Hairy ball thm</u> (see later).

Every smooth $S^2 \ni p \mapsto v(p) \in T_pS^2$ has some zero.

In particular TS^2 more complicated than $S^2 \times \mathbb{R}$

~~$S^2 = B^3 \cup_\varphi B^3$~~

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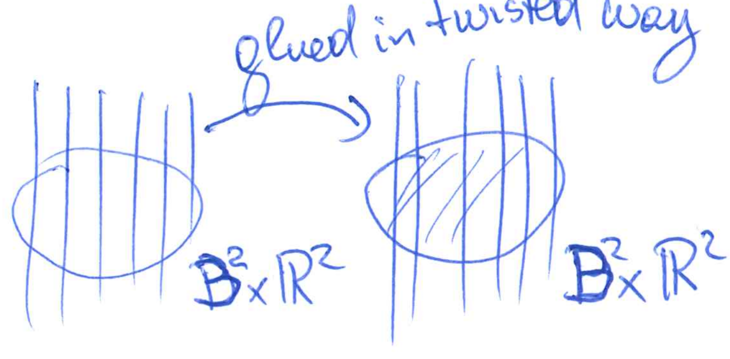


Vector fields

Def: A smooth vector field on M is a smooth function $X : M \longrightarrow TM$, such that $\pi \circ X = id_M$

Note: $\pi(X(p)) = p$ means $X(p) \in T_pM$ (also called "section" of $\pi : TM \to M$)

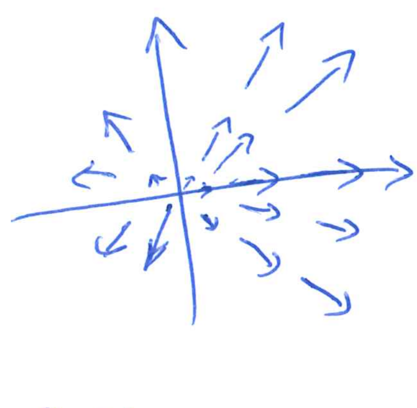
Notation: $\mathfrak{X}(M) := \{ X \mid X \text{ is a smooth vector field on } M \}$

Note: $(U, (x^i))$ chart $\Rightarrow$ Can locally write

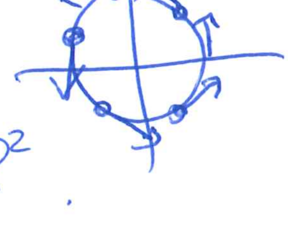
$$X(p) = X^i(p) \frac{\partial}{\partial x^i}\Big|_p$$

In particular X smooth $\Leftrightarrow$ $X^i(p)$ smooth $(i = 1, \dots, n)$

Ex $X = x^i \frac{\partial}{\partial x^i} = r \frac{\partial}{\partial r}$ radial vector field on $\mathbb{R}^n$



Ex $X = \frac{\partial}{\partial \theta}$ rotation vector field on S^1



Ex $X = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$ rotation vf on $\mathbb{R}^2$.

<u>Note:</u> i) X, Y smooth vf $\Rightarrow$ $X+Y$ smooth vf

ii) X smooth vf, f smooth fn $\Rightarrow$ fX smooth vf

$(fX)(p) := f(p)X(p)$

i.e. $\mathfrak{X}(M)$ is a <u>module</u> over the ring $C^\infty(M)$.

<u>Note:</u> i) X smooth vf, f smooth fn $\Rightarrow$ Xf smooth fn

$(Xf)(p) = X_p f$

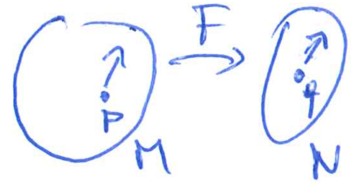
ii) X smooth vf, f, g smooth fns

$$\Rightarrow X(fg) = fXg + gXf$$

i.e. $X : C^\infty(M) \to C^\infty(M)$ is a <u>derivation</u>.

<u>Note:</u> X smooth vf on M, $F: M \to N$ diffeo

$$\Rightarrow (F_* X)_q := dF_{F^{-1}(q)}(X_{F^{-1}(q)})$$



is a smooth vf on N, called the <u>pushforward</u> of X

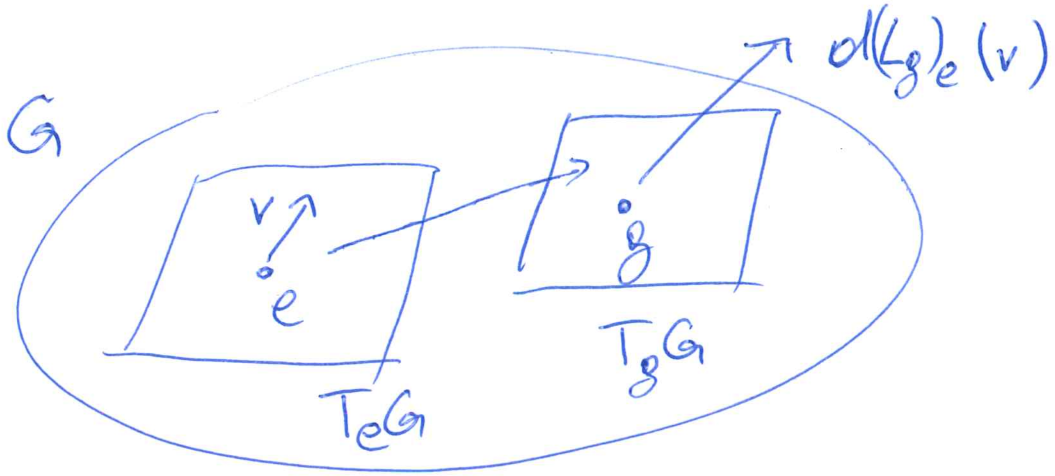
Important example: Lie groups

Recall: A Lie group is a group G that is also a smooth mfd, s.t. $\begin{matrix} G\times G \to G \\ (g,h)\mapsto gh \end{matrix}$ & $\begin{matrix} G \to G \\ g \mapsto g^{-1} \end{matrix}$ are smooth.

Ex $G = GL(n,\mathbb{R}), SO(n), SU(n), T^n, \dots$

Note: $\underbrace{\begin{matrix} L_g : G \to G \\ h \mapsto gh \end{matrix}}_{\text{left multiplication with } g}$ & $\underbrace{\begin{matrix} R_g : G \to G \\ h \mapsto hg \end{matrix}}_{\text{right multiplication with } g}$ are smooth.

Note: $d(L_g)_e : T_eG \to T_gG$



So given any $v \in T_eG \cong \mathbb{R}^n$, we can define a smooth vf $X(g) := d(L_g)_e(v)$

<u>Note:</u> X is a nowhere vanishing smooth vf on G.

This explains why S^1 and $S^3 \cong SU(2)$ are parallelizable.

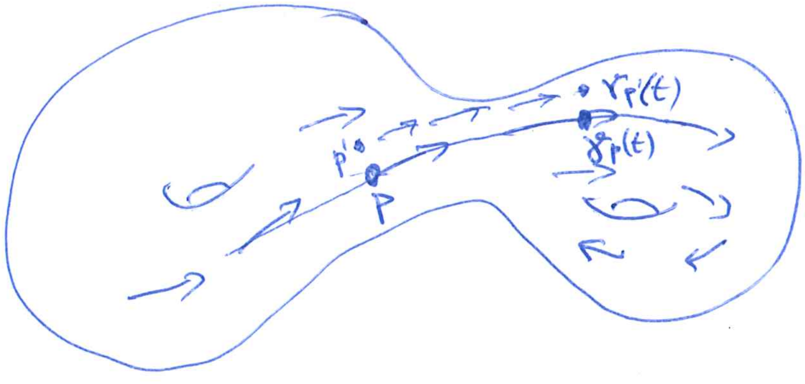
<u>Exer</u> Check that $X(g) := d(L_g)_e(v)$ satisfies $\underbrace{(L_h)_* X = X \quad \forall h \in G}_{\text{"left-invariant vf"}}$

<u>Def</u> $\mathfrak{g} := \{ X \mid X \text{ left-invariant vf on } G \} \cong T_eG$ is called the <u>Lie algebra</u> of G

<u>Note:</u> X, Y left-inv. vfs on $G \Rightarrow [X,Y] := XY - YX$ is again a left-inv. vf.

(use $(L_h)_*[X,Y] = [(L_h)_*X, (L_h)_*Y]$)

Integral curves & flows of vector fields



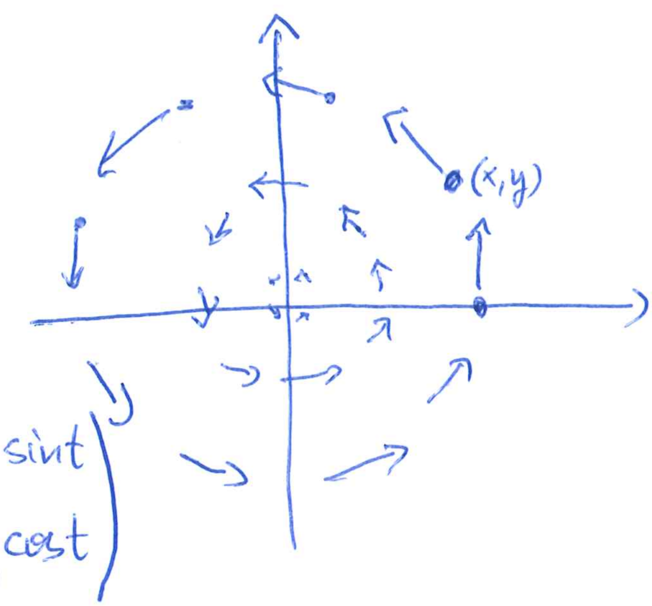
X smooth vector field on M

M smooth mfd

goal: study integral curves $\gamma_p(t)$
& flow $\varphi_t(p) = \gamma_p(t)$

Ex $M = \mathbb{R}^2$, $p = (x, y)$

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$



Solve $\dot{\gamma}(t) = X(\gamma(t))$

get $\gamma_{(x,y)}(t) = \begin{pmatrix} x\cos t - y\sin t \\ x\sin t + y\cos t \end{pmatrix}$

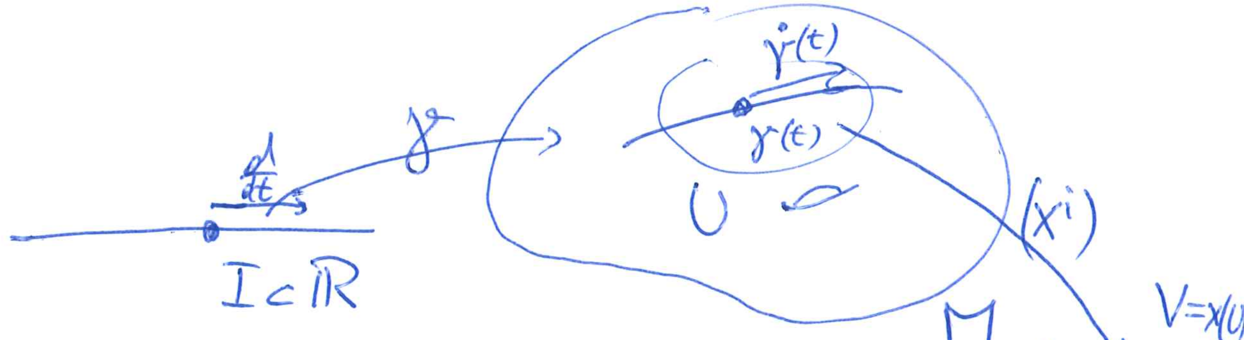
$\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ rotation by angle t.

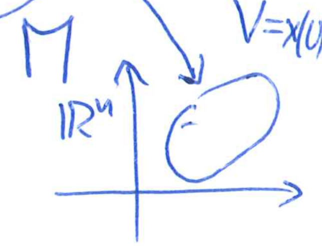
In particular, $\varphi_0 = id$, $\varphi_{s+t} = \varphi_s \circ \varphi_t$.

Def: A smooth curve γ is called an <u>integral curve</u> of a smooth vector field X on a smooth mfd M, if

$$\dot{\gamma}(t) = X(\gamma(t)). \qquad (*)$$

<u>recall:</u>





•) $\dot{\gamma}(t) = d\gamma_t\left(\frac{d}{dt}\right) \in T_{\gamma(t)}M$

•) $M \underset{\pi}{\overset{X}{\rightleftarrows}} TM, \quad \pi \circ X = \mathrm{id}_M$

i.e. concretely $M \ni p \longmapsto X(p) \in T_pM$.

<u>In coordinate chart:</u>

$$X(p) = X^i(p)\frac{\partial}{\partial x^i}\Big|_p, \quad \dot{\gamma}(t) = \dot{\gamma}^i(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

$$(*) \quad \dot{\gamma}^i(t) = X^i(\gamma(t)) \quad (i = 1, \dots, n)$$

Can solve this uniquely, given $\gamma(0) = (\gamma^1(0), \dots, \gamma^n(0))$.

recall from ODE course:

<u>Thm</u> (existence, uniqueness & smooth dependence)

Let X be a smooth vf on $V \subset \mathbb{R}^n$ open.

Then $\forall x_0 \in V$, $\exists$ open nbd $\underset{x_0}{\underset{\Psi}{W}} \subset V$, & $\varepsilon > 0$,

& smooth map
$$\Phi : (-\varepsilon, \varepsilon) \times W \longrightarrow V$$
$$(t, p) \longmapsto \varphi_t(p)$$

such that
$$\begin{cases} \frac{d}{dt} \varphi_t(p) = X(\varphi_t(p)) \\ \varphi_0(p) = p \end{cases}.$$

Moreover if $(W', \varepsilon', \Phi')$ satisfies the same, then $\Phi = \Phi'$ on $\min(\varepsilon, \varepsilon') \times (W \cap W')$.

Cor Let X be a smooth vf on a smooth mfd M.

Then $\exists\ \Omega \subset \mathbb{R}\times M$ open nbd of $\{0\}\times M$
& $\Phi : \Omega \to M$ smooth,
$(t,p) \mapsto \Phi(t,p) \equiv \varphi_t(p)$, st:

(a) $\frac{d}{dt}\varphi_t(p) = X(\varphi_t(p))$
(b) $\varphi_0(p) = p$
$\Big\}\ \forall (t,p)\in\Omega$

(c) $\varphi_s(\varphi_t(p)) = \varphi_{s+t}(p)$ whenever defined.

Moreover, if (Ω', Φ') satisfies the same, then $\Phi = \Phi'$ on $\Omega\cap\Omega'$.

Proof ODE thm $\Rightarrow$ $\exists\ U_i \subset M$ open cover, $\varepsilon_i > 0$,
& $\Phi_i : (-\varepsilon_i, \varepsilon_i)\times U_i \to M$
satisfying (a) – (c).

uniqueness $\Rightarrow$ $\Phi_i = \Phi_j$ on intersection of domains

$\Rightarrow$ get well-defined global map

$$\Phi : \Omega = \bigcup_i \left((-\varepsilon_i, \varepsilon_i)\times U_i\right) \longrightarrow M,$$

which satisfies (a) & (b).

Check (c): $\tau \mapsto \varphi_\tau(\varphi_t(p))$ & $\tau \mapsto \varphi_{t+\tau}(p)$
solve same ODE with initial condition $\varphi_t(p)$ $\Rightarrow$ (c).

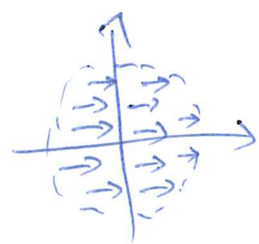
Moreover part, as above. $\square$

Consider $\Omega_{max} = \bigcup \Omega$ maximal domain.

__Def:__ X is __complete__ if it has a globally defined flow, i.e. if $\Omega_{max} = \mathbb{R} \times M$.

__Cor:__ If X is complete, then $\varphi_t : \begin{array}{l} M \to M \\ p \mapsto \Phi(t,p) \end{array}$

is a one-parameter group of diffeomorphisms, i.e. $\varphi_t : M \to M$ diffeos st $\varphi_{s+t} = \varphi_s \circ \varphi_t$.

__Ex__ ·) $X = \frac{\partial}{\partial x}$ on open ball B not complete 

·) $X = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}$ not complete

__Q:__ When is X complete? (Criterion?)

Prop (a) Any smooth vf on a compact mfd is complete.
(b) Any left-invariant vf on a Lie group is complete.

Rmk: More generally: X smooth vf on complete Riemannian mfd,
$|X| \le C \Rightarrow X$ complete.

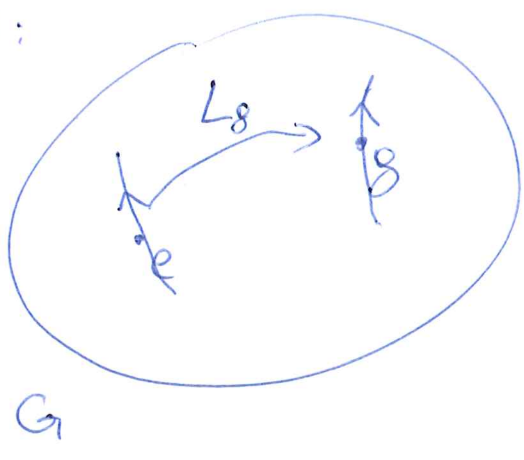
Proof Enough to show that:

$$\exists \varepsilon > 0 : (-\varepsilon, \varepsilon) \times M \subset \Omega_{max} \qquad (*)$$

$$\left(\text{since then } \varphi_t = \underbrace{\varphi_{t/N} \circ \dots \circ \varphi_{t/N}}_{N \text{ times}} \Rightarrow \Omega_{max} = \mathbb{R} \times M \right)$$

$(*)$ holds for (a) by compactness

For (b):



$$L_g : G \to G$$
$$h \mapsto gh$$

γ integral curve of X through e $\Big\} \underset{(L_g)_* X = X}{\Longrightarrow} L_g \circ \gamma$ integral curve of X through g.

$\Rightarrow (*)$ ▨