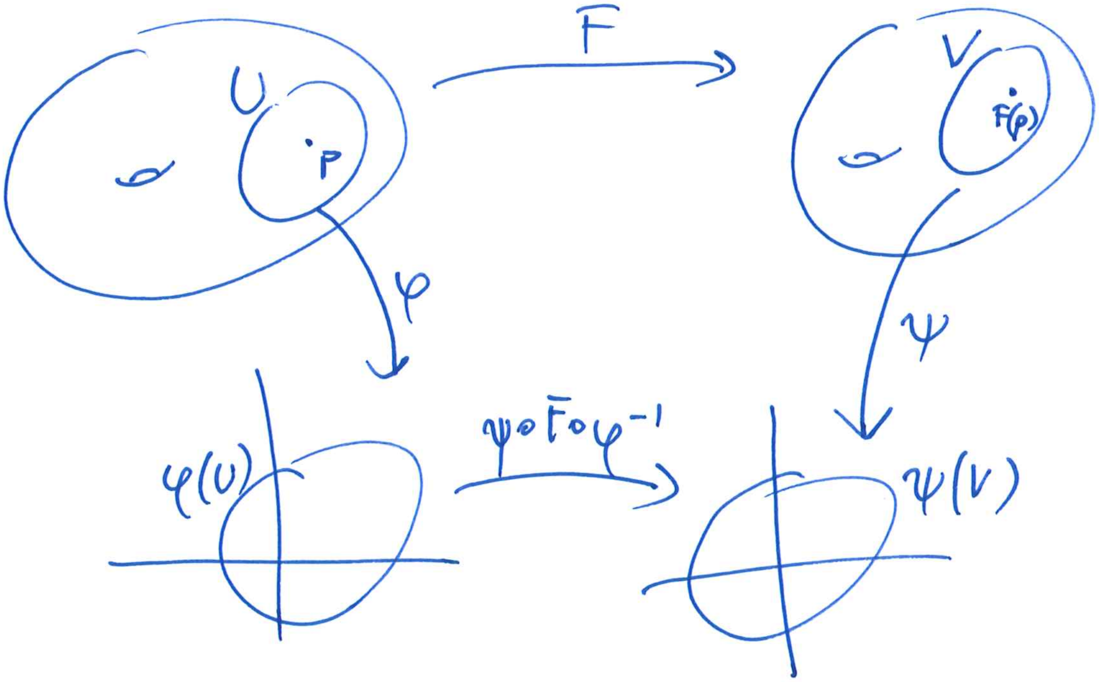
Smooth maps

Def: A continuous map $F: M \to N$ between smooth mfds is called smooth if $\forall p \in M$ $\exists (U, \varphi)$ chart at p & (V, ψ) chart at $F(p)$, such that $\psi \circ F \circ \varphi^{-1}$ is smooth.



Note: smooth in some chart at $p \overset{\text{chain rule}}{\iff}$ smooth in all charts at p.

Notation: $C^\infty(M,N) = \{F: M \to N \mid F \text{ smooth}\}$

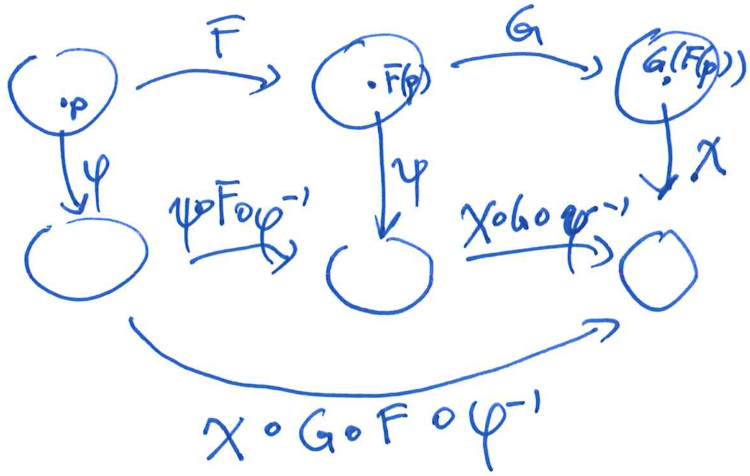
$C^\infty(M) = \{f: M \to \mathbb{R} \mid f \text{ smooth}\}$

Prop (a) $M \xrightarrow{id} M$ smooth

(b) $M \xrightarrow{F} N$ & $N \xrightarrow{G} P$ smooth $\Rightarrow G \circ F : M \to P$ smooth

Proof: (a) clear

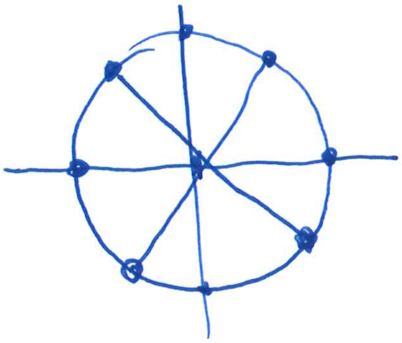
(b)



▨

Hence, $\underbrace{\text{smooth mfds}}_{\text{objects}}$ with $\underbrace{\text{smooth maps}}_{\text{arrows}}$ form a category.

Ex $S^n \xhookrightarrow{\iota} \mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{RP}^n$ is smooth



Def: M & N are diffeomorphic if

$\exists\, F : M \to N$ smooth with smooth inverse.

Ex $B^n \underset{G}{\overset{F}{\rightleftarrows}} \mathbb{R}^n$, $F(x) = \frac{x}{\sqrt{1-|x|^2}}$, $G(y) = \frac{y}{\sqrt{1+|y|^2}}$.

Conj (Poincare) M closed n-dim mfd

M homotopy equivalent to $S^n \Rightarrow M$ homeomorphic to S^n.

true in all dimensions:

$n \geq 5$ Smale

$n = 4$ Freedman

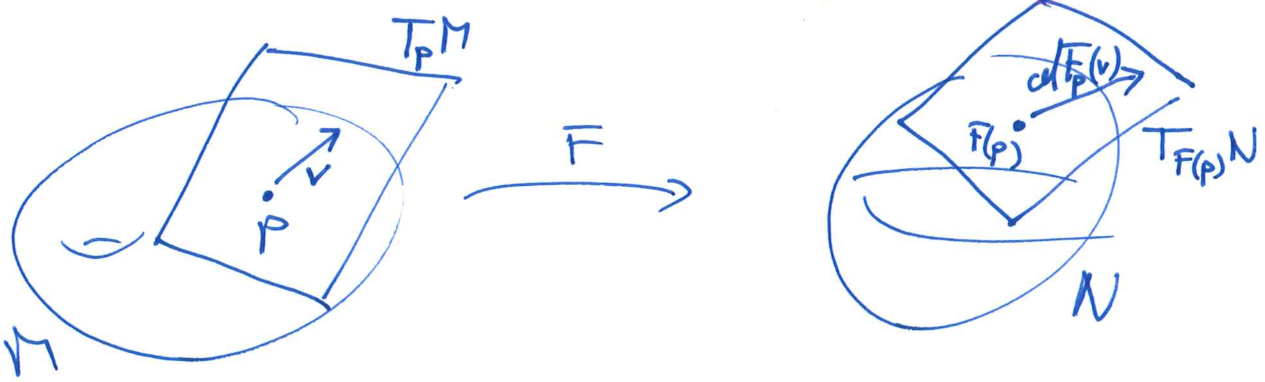
$n = 3$ Perelman

Thm (Milnor) $\exists$ exotic 7-spheres, i.e. M^7 that is homeomorphic but not diffeomorphic to std' S^7.

Open problem: M homeomorphic to $S^4 \overset{?}{\Rightarrow} M$ diffeomorphic to S^4.

Tangent vectors & differential

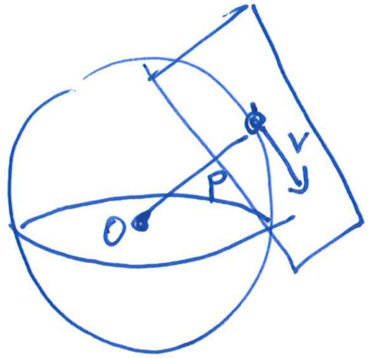
goal: Given $F : M \to N$ smooth map, $p \in M$
want to make sense of linear approximation $dF_p : T_pM \to T_{F(p)}N$



$$T_pM = \{ v \mid v \underbrace{\text{tangent vector}}_{\text{define this first!}} \text{ at } p \}$$

Ex $S^n \subset \mathbb{R}^{n+1}$, $p \in S^n$

$$T_pS^n = \{ v \in \mathbb{R}^{n+1} \mid v \cdot p = 0 \}$$



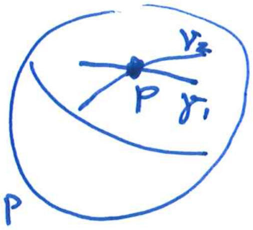
Q: How can we define tangent vectors on general M?

Many approaches:

(i) $M \subset \mathbb{R}^N \rightsquigarrow T_pM \subset T_p\mathbb{R}^N$

⊕ straigthforward ⊖ not intrinsic

(ii) equivalence classes of curves



$$\left.\begin{array}{l}\gamma_1 : I_1 \longrightarrow M \\ \gamma_2 : I_2 \longrightarrow M\end{array}\right. \text{ smooth curves, } \gamma_i(0)=p$$

$$\gamma_1 \sim \gamma_2 :\Leftrightarrow \left.\frac{d}{dt}\right|_{t=0} f(\gamma_1(t)) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma_2(t)) \quad \forall \underset{\text{smooth}}{f} : \overset{\text{nbd of } p.}{\overset{\downarrow}{U}} \to \mathbb{R}$$

$$T_pM = \{[\gamma]\}$$

⊕ geometric ⊖ vector space structure not manifest

(iii) directional derivative

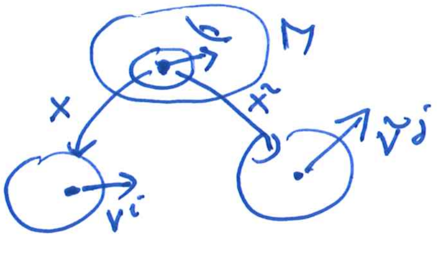
$$"v \equiv D_v" \qquad D_v(fg) = f D_v g + g D_v f$$

i.e. tangent vector = linear differential operator that satisfies the product rule

⊕ intrinsic, vector space ⊖ abstract

(iv) using charts

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i} v^i$$



⊕ concrete

⊖ transformati
rules.

Tangent vectors (using approach (iii))

Def: A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a derivation at p if it satisfies

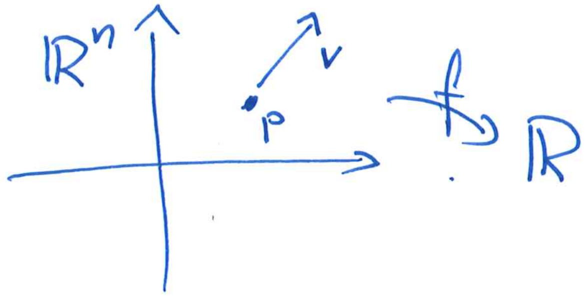
$$v(fg) = f(p)\,v(g) + g(p)\,v(f) \quad \forall f, g \in C^\infty(M)$$

Idea: "$v(f) \equiv (D_v f)|_p$"

Def: $T_pM := \{v \mid v \text{ is a derivation at } p\}$ the tangent space at p.

Note: T_pM is a vector space.

Q: $T_p\mathbb{R}^n = ?$



For any $v \in \mathbb{R}^n$ can define

$$D_v|_p f := \frac{d}{dt}\Big|_{t=0} f(p+tv) \quad , \quad f \in C^\infty(\mathbb{R}^n)$$

directional derivative at p in direction v.

It satisfies $D_v|_p(fg) = f(p)\,D_v|_p(g) + g(p)\,D_v|_p(f)$ ✓

i.e. $D_v|_p$ is a derivation at p

Prop: The map $\mathbb{R}^n \to T_p\mathbb{R}^n$

$v \longmapsto D_v|_p$ is an isomorphism.

Proof: ·) clearly linear ✓

·) injective: suppose $D_v|_p(f) = 0 \ \forall f \in C^\infty(\mathbb{R}^n)$

write $v = (v^1, \dots, v^n) \in \mathbb{R}^n$

apply to $f = x^j : \mathbb{R}^n \to \mathbb{R}$ the j-th coordinate fn.

$$\Rightarrow \underline{\underline{0}} = D_v|_p(x^j) = \frac{d}{dt}\Big|_{t=0}(p^j + tv^j) = \underline{\underline{v^j}}$$

·) surjective: wlog $p = 0$

given $w \in T_0\mathbb{R}^n$ define $v^j := w(x^j)$

want to show $w = D_v|_0$

$$\text{Write } f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)\,dt = \sum_{j=1}^n x^j \underbrace{\int_0^1 \partial_j f(tx)\,dt}_{=: f_j(x)}$$

$$\Rightarrow \left\{ \begin{array}{l} w(f) = \sum_{j=1}^n w(x^j) f_j(0) \\ D_{v|0}(f) = \sum_{j=1}^n D_{v|0}(x^j) f_j(0) \end{array} \right.))$$

$\square$

Cor: $D_{e_i}|_p \equiv \frac{\partial}{\partial x^i}\Big|_p$ is a basis for $T_p\mathbb{R}^n$.

The differential

$F : M \to N$ smooth, $p \in M$

Def: $dF_p : T_pM \to T_{F(p)}N$, $dF_p(v)(f) := v(f \circ F)$

$\forall f \in C^\infty(N)$

Note:

- $f \circ F \in C^\infty(M)$, so $v(f \circ F)$ makes sense.
- $$dF_p(v)(fg) = v((fg)\circ F) = v((f\circ F)(g \circ F))$$
  $$= f(F(p))\, \underbrace{v(g\circ F)}_{= dF_p(v)(g)} + g(F(p))\, \underbrace{v(f \circ F)}_{= dF_p(v)(f)}$$
  i.e. $dF_p(v)$ is indeed a derivation at $F(p)$.

Basic properties:

(i) $dF_p : T_pM \to T_{F(p)}N$ is linear

(ii) $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ (chain rule)

In particular, $(dF_p)^{-1} = d(F^{-1})_{F(p)}$ for F diffeo.

Proof: (i) clear ✓

(ii)
$$\begin{tikzcd}
M \ni p \arrow[r, "F"] & N \ni F(p) \arrow[r, "G"] \arrow[dr, bend right, "f\circ G"'] & P \ni G(F(p)) \arrow[d, "f"] \\
 & & \mathbb{R}
\end{tikzcd}$$

$v \in T_pM$

$$V(f\circ G\circ F) \underset{\substack{\uparrow \\ \text{def of } dF}}{=} dF_p(v)(f\circ G) \underset{\substack{\uparrow \\ \text{def of } dG}}{=} dG_{F(p)}(dF_p(v))(f)$$ ▨

Lemma (locality) Given $p\in M$.

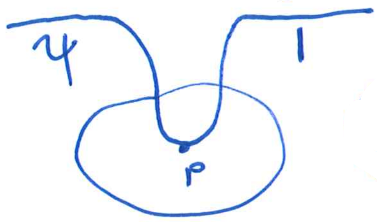
If $f,g\in C^\infty(M)$ agree in some nbd of p

then $vf = vg \quad \forall v\in T_pM$

Proof $h:=f-g$ vanishes in nbd of p.

choose $\psi\in C^\infty(M)$

such that



$$\psi\equiv 1 \text{ on support of } h, \quad \psi(p)=0$$

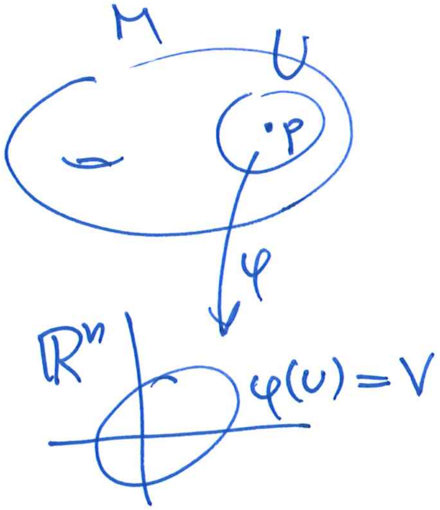
$$\Rightarrow \quad h=\psi h$$

$$\Rightarrow \quad v(h) = v(\psi h) = \underbrace{\psi(p)}_{=0} v(h) + \underbrace{h(p)}_{=0} V(\psi) = 0$$ ▨

Cor M n-dim smooth mfd, $p \in M$

$\Rightarrow$ T_pM is an n-dim vectorspace.

Proof

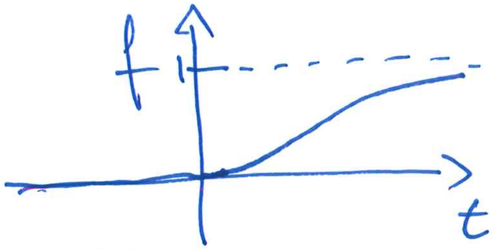


$$T_pM \underset{\uparrow \text{ locality}}{\cong} T_pU \underset{d\varphi_p \text{ is iso}}{\cong} T_{\varphi(p)}V \underset{\text{locality}}{\cong} T_{\varphi(p)}\mathbb{R}^n$$

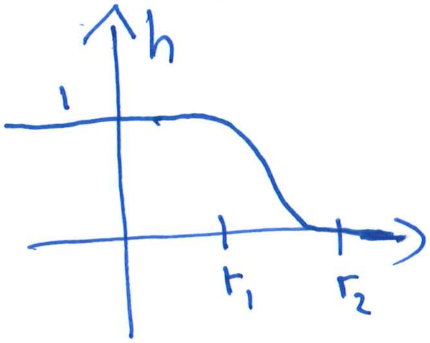
□

Q: Existence of bump function ψ?

Note: $f(t) := \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \le 0 \end{cases}$ is smooth



$$h(t) := \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}$$

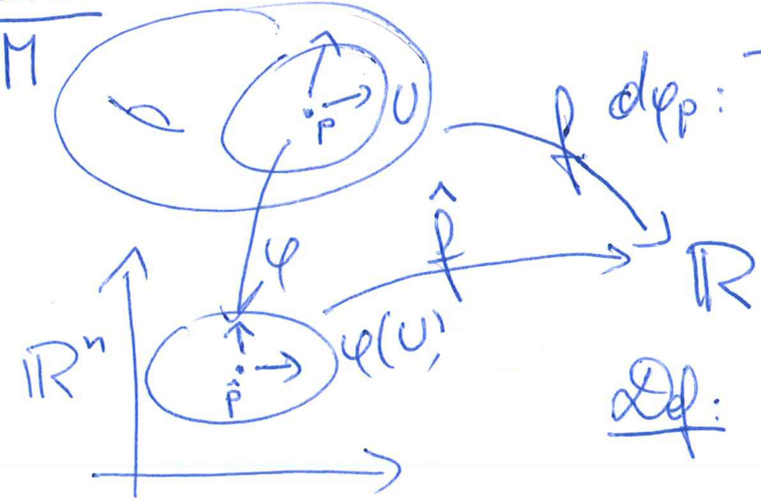


Computations in local coordinates

<u>recall:</u> $T_pM = \{ v \mid v \text{ derivation at } p \}$, $T_p\mathbb{R}^n = \text{span}\{\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\}$

$$dF_p : T_pM \to T_{F(p)}N \,, \quad dF_p(v)(f) = v(f \circ F)$$

<u>in a chart:</u>



$$d\varphi_p : T_pM \xrightarrow{\cong} T_{\varphi(p)}\mathbb{R}^n$$
$$\qquad\qquad\qquad\qquad \cup\!\!\!\!\shortmid$$
$$\frac{\partial}{\partial x^1}\Big|_{\varphi(p)}, \dots, \frac{\partial}{\partial x^n}\Big|_{\varphi(p)}$$

<u>Def:</u> $\frac{\partial}{\partial x^i}\Big|_p := (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right)$

So $\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p$ is a basis of T_pM.

<u>Q:</u> How does $\frac{\partial}{\partial x^i}\big|_p$ act on $f \in C^\infty(U)$?

<u>Compute:</u>

$$\underline{\underline{\frac{\partial}{\partial x^i}\Big|_p f}} = (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right) f = d(\varphi^{-1})_{\varphi(p)}\left(\frac{\partial}{\partial x^i}\Big|_{\varphi(p)}\right) f$$

$$= \frac{\partial}{\partial x^i}\Big|_{\varphi(p)} (f \circ \varphi^{-1}) = \underline{\underline{\frac{\partial \hat{f}}{\partial x^i}(\hat{p})}} \,,$$

where $\underbrace{\hat{f} = f \circ \varphi^{-1}}_{\text{"coordinate representation of } f \text{ in the coord system } \varphi = (x^i)}$ and $\hat{p} = \varphi(p)$

Hence, can write any $v \in T_pM$ uniquely as

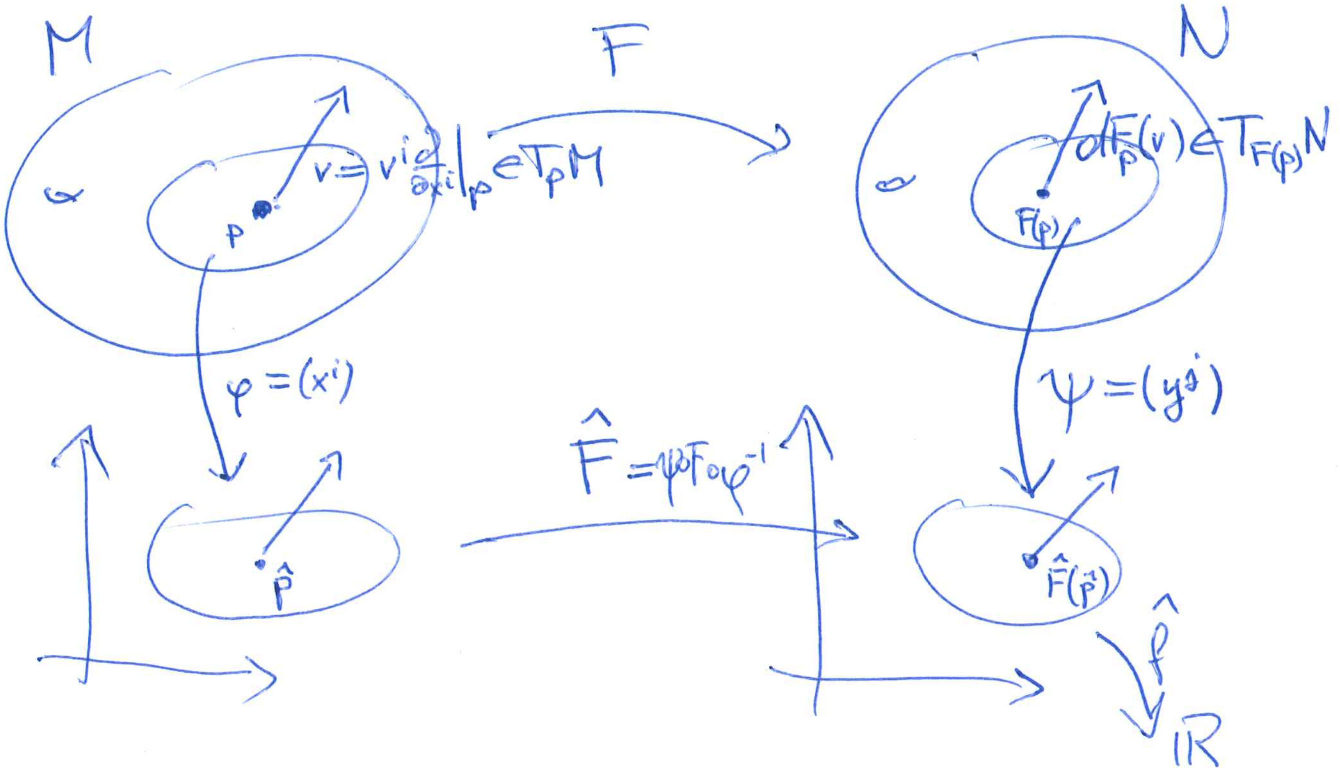
$$v = v^i \frac{\partial}{\partial x^i}\Big|_p$$

(Sum convention: $a^i b_i \equiv \sum_{i=1}^{n} a^i b_i$)

Q: What are the coefficients v^i?

Compute: $\underline{\underline{v(x^j)}} = v^i \frac{\partial}{\partial x^i}\Big|_p (x^j) = \underline{\underline{v^j}}$.

Q: Given $F: M \to N$ smooth, $p \in M$, $v \in T_pM$, What is $dF_p(v)$ in local coordinates?



Compute:

$$dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = dF_p\left(d(\varphi^{-1})_{\hat{p}}\left(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}\right)\right)$$

$$= d(\psi^{-1})_{\hat{F}(\hat{p})}\left(d\hat{F}_{\hat{p}}\left(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}\right)\right)$$

Now, $d\hat{F}_{\hat{p}}\left(\frac{\partial}{\partial x^i}\Big|_{\hat{p}}\right)\hat{f} = \frac{\partial}{\partial x^i}\Big|_{\hat{p}}(\hat{f}\circ\hat{F})$

$$\underset{\substack{\uparrow\\ \text{chain rule in } \mathbb{R}^n}}{=} \frac{\partial \hat{f}}{\partial y^j}(\hat{F}(\hat{p}))\,\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})$$

$$= \left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\,\frac{\partial}{\partial y^j}\Big|_{\hat{F}(\hat{p})}\right)\hat{f}$$

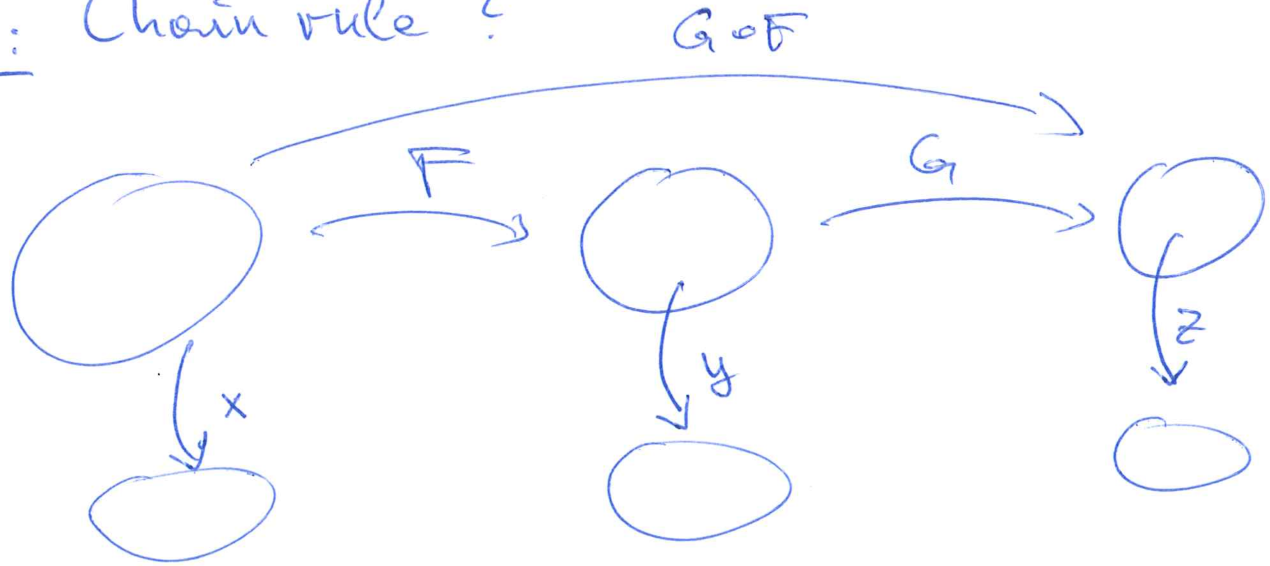
Thus, $\underline{\underline{dF_p\left(\frac{\partial}{\partial x^i}\Big|_p\right)}} = d(\psi^{-1})_{\hat{F}(\hat{p})}\left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\,\frac{\partial}{\partial y^j}\Big|_{\hat{F}(\hat{p})}\right)$

$$= \underline{\underline{\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\,\frac{\partial}{\partial y^j}\Big|_{F(p)}}}$$

i.e. dF_p is given by the Jacobi matrix.

(I'd simply write $dF\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial F^j}{\partial x^i}\frac{\partial}{\partial y^j}$)

Q: Chain rule ?



Compute:

$$\underline{\underline{d(G\circ F)\left(\frac{\partial}{\partial x^i}\right)}} = dG\left(dF\left(\frac{\partial}{\partial x^i}\right)\right)$$

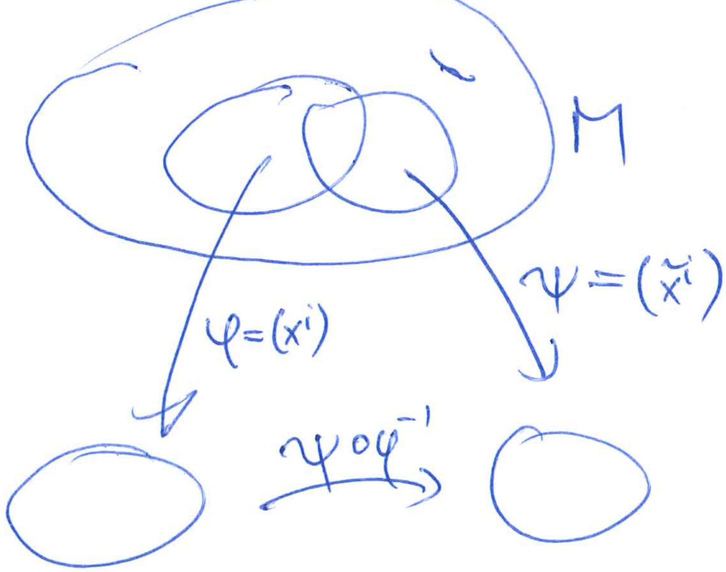
$$= dG\left(\frac{\partial F^j}{\partial x^i}\frac{\partial}{\partial y^j}\right)$$

$$= \frac{\partial F^j}{\partial x^i}\, dG\left(\frac{\partial}{\partial y^j}\right)$$

$$= \underline{\underline{\frac{\partial F^j}{\partial x^i}\frac{\partial G^k}{\partial y^j}\frac{\partial}{\partial z^k}}}$$

i.e. simply multiply Jacobi matrices.

Q: Change of coords?



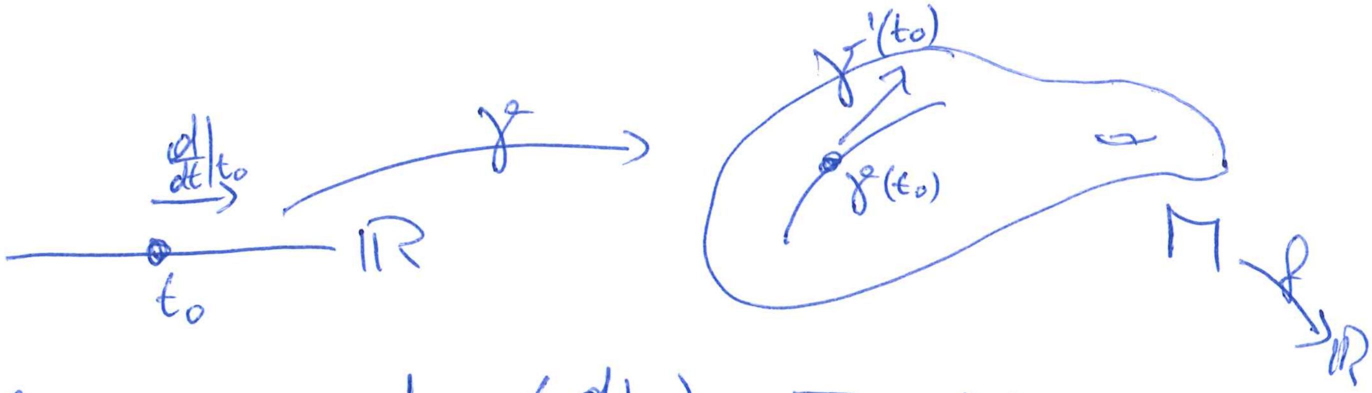
$$\psi \circ \varphi^{-1}(x) = (\tilde{x}^1(x), \ldots, \tilde{x}^n(x))$$

$$\Rightarrow \begin{cases} \left.\dfrac{\partial}{\partial x^i}\right|_p = \dfrac{\partial \tilde{x}^j}{\partial x^i} \left.\dfrac{\partial}{\partial \tilde{x}^j}\right|_p \\ \tilde{V}^j = \dfrac{\partial \tilde{x}^j}{\partial x^i} V^i \end{cases}$$

Ex $(x, y) = (r\cos\theta, r\sin\theta)$

$$\Rightarrow \begin{cases} \partial_r = \cos\theta\,\partial_x + \sin\theta\,\partial_y \\ \partial_\theta = -r\sin\theta\,\partial_x + r\cos\theta\,\partial_y \end{cases}$$

Velocity vector of curves



Def: $\gamma'(t_0) := d\gamma_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{\gamma(t_0)}M$

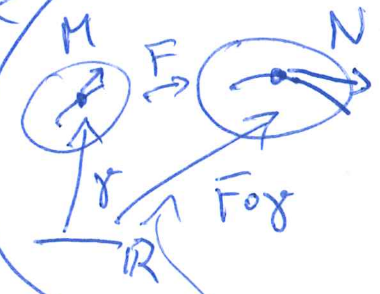
is called the velocity vector of γ at time t_0.

Note: $\underline{\underline{\gamma'(t_0)f}} = d\gamma_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right)f = \frac{d}{dt}\Big|_{t_0}(f\circ\gamma) = \underline{\underline{(f\circ\gamma)'(t_0)}}$

Q: $\gamma'(t)$ in local coords?

A: Writing $\gamma(t) = (\gamma^i(t))$ we have

$$\underline{\underline{\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0)\frac{\partial}{\partial x^i}\Big|_{\gamma(t_0)}}}$$



In particular, given $v = v^i\frac{\partial}{\partial x^i}\Big|_p \in T_pM$, $\varphi(p) = 0$

$\gamma(t) = (tv^1, \dots, tv^n)$ satisfies $\gamma'(0) = v$.

Finally, $\boxed{dF_p(v) = (F\circ\gamma)'(0)}$ for any smooth γ with $\gamma(0) = p$, $\gamma'(0) = v$.