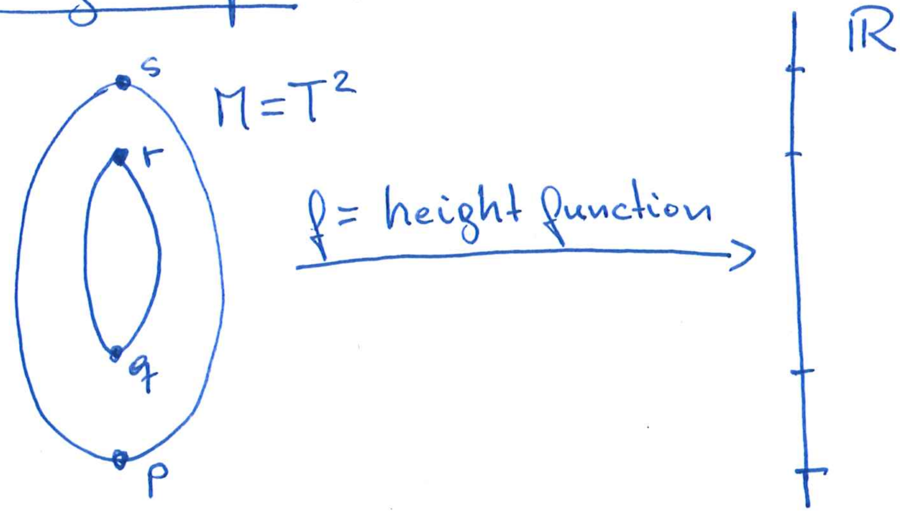
Morse theory & applications

Motivating example

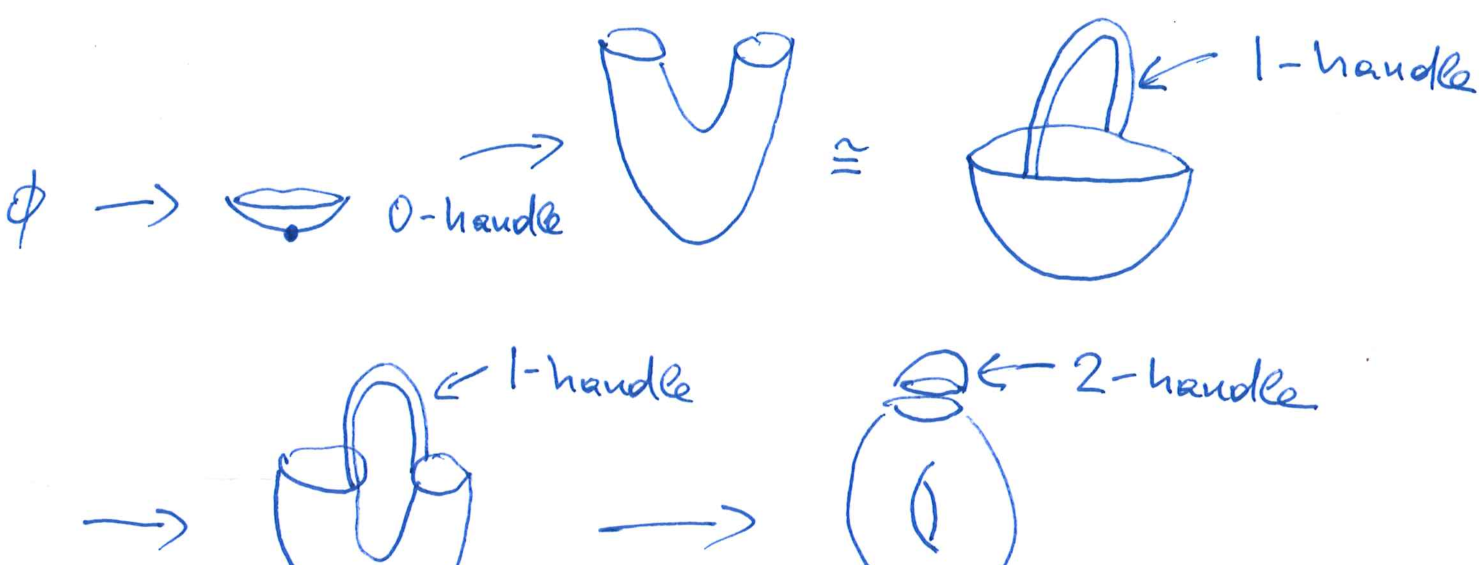


4 critical points: p, $\underbrace{q, r}_{\text{index } 1\ (f = x^2 - y^2)}$, s ← index 2 ($f = -x^2 - y^2$)

index 0 ($f = x^2 + y^2$)

$$M_a := \{ x \in M \mid f(x) \leq a \}$$

Increase a $\rightsquigarrow$ topology changes by attaching handles



Def A smooth function $f: M \to \mathbb{R}$ is called a **Morse function** closed smooth mfd
if all its critical points are nondegenerate, i.e. whenever $df_p = 0$ then $\det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) \neq 0$.

Note: Morse $\Rightarrow$ isolated crit. pts $\underset{M \text{ compact}}{\Rightarrow}$ finitely many crit. pts

Prop Morse functions are generic.

Proof •) clearly stable ✓

•) dense: Consider first $g: U \to \mathbb{R}$ smooth on $U \subset \mathbb{R}^n$ open.

Set $g_b := g + b_1 x_1 + \ldots + b_n x_n$

Note that: p crit pt of $g_b \Leftrightarrow d_p g = -b$

and $\operatorname{Hess} g_b = \operatorname{Hess} g$

Thus: g_b Morse $\Leftrightarrow -b$ is regular value of $(x_1, \ldots, x_n) \mapsto \left(\frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}\right)$

Sard $\Rightarrow g_b$ Morse for almost every b.

Now $M^n \hookrightarrow \mathbb{R}^N$. Set $f_a := f + a_1 x_1 + \ldots + a_N x_N$.

Near any $p \in M$ $\exists\, i_1, \ldots, i_n$ st $(x_{i_1}, \ldots, x_{i_n})$ are coords near p.

After relabling $(x_1, \ldots, x_n): U \xrightarrow{\cong} V \subset \mathbb{R}^n$ coords near p.

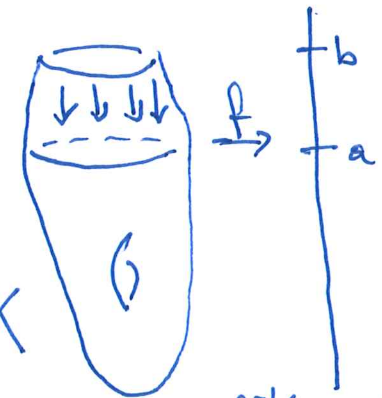
Set $f_{(0,c)} = f + c_{n+1} x_{n+1} + \ldots + c_N x_N$. Write $a = (b, c)$

By the above $(f_{(0,c)})_b$ is Morse for almost every b.

$\Rightarrow f_a$ is Morse for almost every a. $\boxed{//}$

Thm 1 (Morse) If $f: M \to \mathbb{R}$ is Morse and $[a,b]$ contains no critical points, then $M_a \underset{\uparrow}{\cong} M_b$
diffeomorphic.

Proof Let g be a Riem. metric on M.



Consider the vectorfield $\operatorname{grad} f$ defined by $\langle \operatorname{grad} f, X \rangle_g = X(f) \;\; \forall X$

Choose $\rho: M \to \mathbb{R}$ smooth st $\rho = \frac{1}{|\operatorname{grad} f|^2}$ on $f^{-1}([a,b])$

Let φ_t be the flow of $\rho \cdot \operatorname{grad} f$.

If $\varphi_t(p) \in f^{-1}([a,b])$ then

$$\frac{d}{dt} f(\varphi_t(p)) = \langle \operatorname{grad} f, \underbrace{\tfrac{d}{dt} \varphi_t(p)}_{= \rho \cdot \operatorname{grad} f} \rangle = 1.$$

Thus $t \mapsto f(\varphi_t(p))$ is linear with derivative $+1$ as long as $f(\varphi_t(p))$ lies between a & b.

So φ_{b-a} is a diffeo that carries M_a to M_b $\boxtimes$

Thm 2 (Morse) If $f^{-1}([a,b])$ contains a unique nondegenerate critical point in its interior, which has index k, then M_b is obtained from M_a by attaching a k-handle, namely $M_b \cong M_a \cup_{\partial D^k \times D^{n-k}} (D^k \times D^{n-k})$

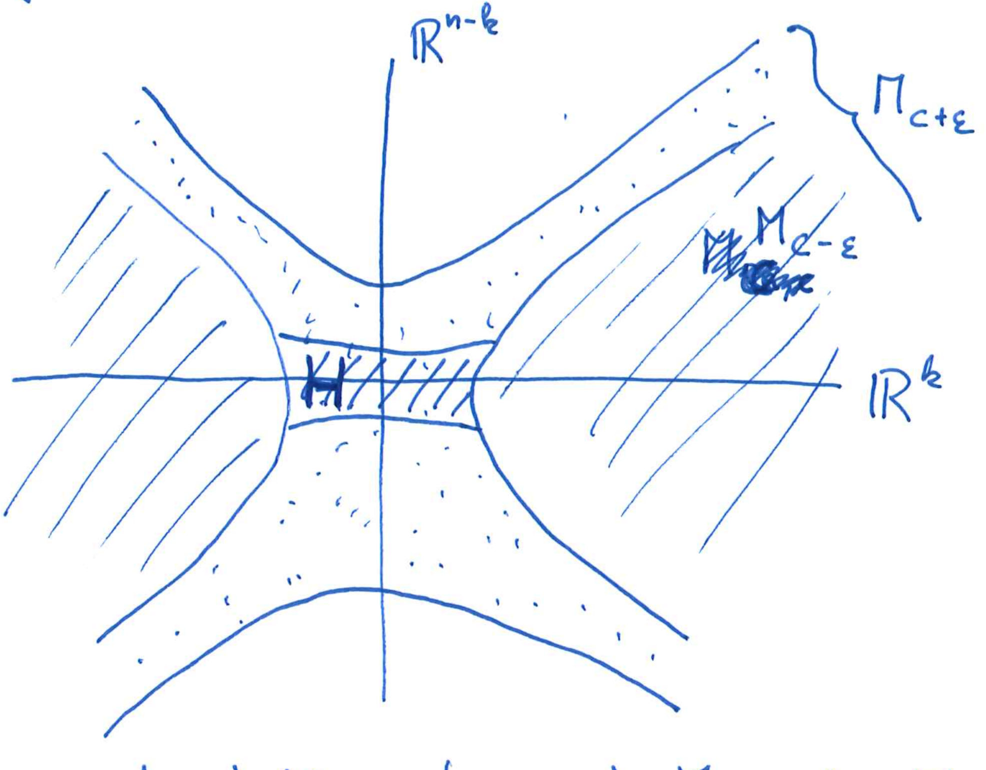
$\uparrow$ homeomorphic (or diffeomorphic after smoothing corners)

Proof (sketch)

Near the critical point p can choose coords st

$$f(x) = f(p) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$$

So

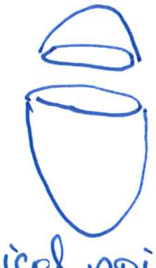


$c := f(p) \in (a,b)$
$\varepsilon > 0$ small

Use gradient flow to push $M_{c+\varepsilon}$ to $M_{c-\varepsilon} \cup_{\partial H} H$

Together with Thm 1 this implies the assertion. $\boxtimes$

Cor (Reeb) If a closed smooth n-mfd admits a Morse function with exactly 2 critical points then it is homeomorphic to S^n.



Rmk Pushing these ideas further (cancelling critical points) Smale proved the Poincare conjecture for $n \geq 5$.

Cor Every closed smooth mfd admits a handle decomposition (in part. is a finite CW-complex)

Cor (Morse inequalities) If $f : M \to \mathbb{R}$ is Morse, then

$$\#\{\text{crit. pts of } f \text{ of index } k\} \geq \dim H^k(M)$$

Prof WLog f has critical points with distinct critical values. Let $a_0 < \dots < a_N$ st $f(M) \subset (a_0, a_N)$ and each $[a_{i-1}, a_i]$ contains 1 crit. value

We will prove by induction that

$$\#\{\text{crit pts of } f|_{M_{a_i}} \text{ of index } k\} \geq \dim H^k(M_{a_i})$$

Clearly true for $M_{a_0} = \phi$.

By Thm 2, we have

$$M_{a_i} \cong M_{a_{i-1}} \underset{\partial D^k \times D^{n-k}}{\cup} (D^k \times D^{n-k})$$

Apply Mayer-Vietoris with

$$U = \mathrm{int}(D^k) \times D^{n-k}$$

$$V = M_{a_{i-1}} \underset{\partial D^k \times D^{n-k}}{\cup} (D^k \setminus D^k_{1/2} \times D^{n-k})$$

Note that $U \simeq *$, $V \simeq M_{a_{i-1}}$, $U \cap V \simeq S^{k-1}$

Thus get

$$H^j(M_{a_i}) \xrightarrow{\cong} H^j(M_{a_{i-1}}) \text{ for } j \neq k, k-1$$

and for $j = k, k-1$ still get either

$$\begin{cases} \dim H^k(M_{a_i}) = \dim H^k(M_{a_{i-1}}) + 1 \\ \dim H^{k-1}(M_{a_i}) = \dim H^{k-1}(M_{a_{i-1}}) \end{cases}$$

or $$\begin{cases} \dim H^k(M_{a_i}) = \dim H^k(M_{a_{i-1}}) \\ \dim H^{k-1}(M_{a_i}) = \dim H^{k-1}(M_{a_{i-1}}) - 1. \end{cases}$$

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