Mayer-Vietoris computations

<u>Idea:</u> $M = U \cup V$

Compute $H^*(M)$ out of $H^*(U)$, $H^*(V)$ and $H^*(U\cap V)$

(similar strategy: Seifert-Van Kampen for π_1)

<u>Setup:</u> M smooth mfd, $U, V \subset M$ open s.t. $M = U \cup V$

recall for any $W \subset M$ open have restriction map

$$\Omega^k(M) \longrightarrow \Omega^k(W)$$
$$\omega \longmapsto \omega|_W := i_W^* \omega, \text{ where } i_W : W \hookrightarrow M.$$

So this gives rise to

$$\Omega^k(M) \xrightarrow{R} \Omega^k(U) \oplus \Omega^k(V)$$
$$\omega \longmapsto (\omega|_U, \omega|_V)$$

and $\Omega^k(U) \oplus \Omega^k(V) \xrightarrow{D} \Omega^k(U \cap V)$

$$(\omega, \eta) \longmapsto \omega|_{U\cap V} - \eta|_{U\cap V}$$

<u>Prop</u> (Mayer-Vietoris)

$$0 \to \Omega^*(M) \xrightarrow{R} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{D} \Omega^*(U\cap V) \to 0$$

is a short exact sequence of differential complexes.

Proof: ·) R injective, since $\omega \in \Omega^k(M)$ uniquely determined by $\omega|_U$ & $\omega|_V$.

·) $\mathrm{im}(R) = \ker(D)$, since $\eta \in \Omega^k(U)$, $\nu \in \Omega^k(V)$ can be glued together to a global form $\omega \in \Omega^k(M)$

$$\iff \eta|_{U\cap V} = \nu|_{U\cap V}.$$

·) to show that D is surjective, choose $\rho_U, \rho_V : M \to [0,1]$ smooth st $\rho_U + \rho_V = 1$, $\mathrm{spt}(\rho_U) \subset U$, $\mathrm{spt}(\rho_V) \subset V$.

Given $\omega \in \Omega^k(U\cap V)$ observe that

$$\rho_V\, \omega \in \Omega^k(U)$$
$$\rho_U\, \omega \in \Omega^k(V)$$

where we extended by 0.

So $\omega = D(\rho_V \omega, -\rho_U \omega)$. ☒

Cor (Mayer-Vietoris) There is a long exact sequence

$$\cdots \to H^k(U) \oplus H^k(V) \to H^k(U\cap V) \xrightarrow{\delta} H^{k+1}(M) \to H^{k+1}(U) \oplus H^{k+1}(V) \to \cdots$$

Proof: This follows immediately from the prop & the zig-zag lemma. ☒

Note: $\delta([\omega])$ is represented by the $k+1$ form

$$d(\rho_V \omega) = -d(\rho_U \omega),$$ which is supported in $U \cap V$.

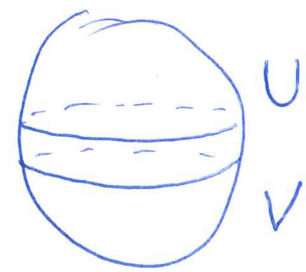
Applications

Prop $H^k(S^n) = \begin{cases} \mathbb{R} & \text{if } k = 0, n \\ 0 & \text{else} \end{cases}$

Proof $\underline{n=1:}$ ✓

$\underline{n-1 \rightsquigarrow n:}$ Choose $U := S^n \cap \{x_{n+1} > -\varepsilon\}$

$V := S^n \cap \{x_{n+1} < \varepsilon\}$



Then $U \cong \mathbb{R}^n$, $V \cong \mathbb{R}^n$, $U \cap V \underset{\substack{\uparrow \\ \text{diffeomorphic}}}{\cong} S^{n-1} \times \mathbb{R} \underset{\substack{\uparrow \\ \text{homotopy equivalent}}}{\simeq} S^{n-1}$

For $k > 1$, Mayer-Vietoris gives

$$0 = H^{k-1}(U) \oplus H^{k-1}(V) \longrightarrow H^{k-1}(S^{n-1}) \longrightarrow H^k(S^n) \longrightarrow H^k(U) \oplus H^k(V) = 0$$

so $H^{k-1}(S^{n-1}) \cong H^k(S^n)$ as long as $k > 1$.

And for $k = 0, 1$ it yields

$$0 \to H^0(S^n) \cong \mathbb{R} \hookrightarrow H^0(U) \oplus H^0(V) \cong \mathbb{R}^2 \twoheadrightarrow H^0(S^{n-1}) = \mathbb{R} \xrightarrow{0} H^1(S^n) \longrightarrow 0$$

$\Rightarrow H^1(S^n) = 0$ $\boxed{/\!/}$

Cor (Brouwer FPT again) $\nexists\, r : D^n \to S^{n-1}$

Proof
$$\begin{tikzcd} S^{n-1} \arrow[rr, "id"] \arrow[dr, hook] & & S^{n-1} \\ & D^n \arrow[ur, "r"'] & \end{tikzcd} \quad \underset{\substack{\text{apply} \\ H^{n-1} \text{ functor}}}{\Longrightarrow} \quad \begin{tikzcd} \mathbb{R} & & \mathbb{R} \arrow[ll, "id"'] \arrow[dl] \\ & 0 \arrow[ul] & \end{tikzcd}$$

$\lightning$ $\boxed{/\!/}$

<u>Prop</u> $H^k(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & \text{if } 0 \le k \le 2n \text{ is even} \\ 0 & \text{else} \end{cases}$

<u>Proof:</u> <u>$n=1$:</u> $\mathbb{CP}^1 \cong S^2$ ✓

<u>$n-1 \rightsquigarrow n$:</u> Let $U := \{[z_0 : \ldots : z_n] \in \mathbb{CP}^n : |z_0|^2 + \ldots + |z_{n-1}|^2 > |z_n|^2\}$

Scale z_n by $t \in [0,1]$ $\Rightarrow$ $U \simeq \mathbb{CP}^{n-1}$

Let $V := \{[z_0 : \ldots : z_n] \in \mathbb{CP}^n : z_n \neq 0\}$

Scale $z_0, \ldots, z_{n-1}$ by t $\Rightarrow$ $V \simeq pt$

Now $U \cap V = \{[z_0 : \ldots : z_{n-1} : 1] : |z_0|^2 + \ldots + |z_{n-1}|^2 > 1\}$

$\simeq \{[z_0 : \ldots : z_{n-1} : 1] : |z_0^2|^2 + \ldots + |z_{n-1}|^2 = 2\}$

$\cong S^{2n-1}$.

Already know $H^0(\mathbb{CP}^n) \cong \mathbb{R}$, $H^{2n}(\mathbb{CP}^n) \cong \mathbb{R}$.

For $0 < k < 2n$ Mayer-Vietoris gives $H^{2n-1}(\mathbb{CP}^n) = 0$ and

$$H^k(\mathbb{CP}^n) \xrightarrow{\cong} H^k(\mathbb{CP}^{n-1})$$

$\boxed{/\!/}$

<u>Rmk</u> Using Poincare duality (see next lecture) one can show that in fact

$$H^*(\mathbb{CP}^n) \cong \mathbb{R}[x]/(x^{n+1})$$

as a ring / graded-algebra.

Prop If M is compact, then $\dim H^k(M) < \infty$.

Proof For $U, V \subset M$ open from the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^{k-1}(U \cap V) \xrightarrow{\delta} H^k(U \cup V) \xrightarrow{R} H^k(U) \oplus H^k(V) \to \cdots$$

we get $H^k(U \cup V) \cong \ker R \oplus \operatorname{im} R \cong \operatorname{im} \delta \oplus \operatorname{im} R$.

Thus, if $H^k(U)$, $H^k(V)$ & $H^{k-1}(U \cap V)$ are finite dim, then so is $H^k(U \cup V)$.

Hence, by induction on covering, the prop. follows from belows lemma. $\boxtimes$

Lemma: Every cpt smooth mfd admits a finite good cover, namely $M = U_1 \cup \ldots \cup U_k$ st $U_{i_1} \cap \ldots \cap U_{i_j} \cong \mathbb{R}^n$ whenever nonempty.

To prove the lemma, we need a bit of geometry:

Def: A Riemannian metric g is a smooth section of $T^*M \otimes T^*M$ that is symmetric & positive definite.

Note: Whitney $\Rightarrow$ $M \hookrightarrow \mathbb{R}^N$, so setting $g_p(X, Y) := \langle X, Y \rangle_{\mathbb{R}^N}$ we see that every smooth mfd can be equipped with a Riem. metric. $(X, Y \in T_pM)$.

In particular, $d_g(x,y) := \inf \left\{ \int_0^1 \sqrt{\langle \dot\gamma(t), \dot\gamma(t) \rangle_g} \;\middle|\; \begin{array}{l} \gamma : [0,1] \to M \\ \gamma(0) = x,\ \gamma(1) = y \end{array} \right\}$

is a distance, which is realized by minimizing geodesics.

Proof of lemma: Choosing $r > 0$ small enough, the geodesic balls $B(p,r) := \{ q \in M : d_g(p,q) < r \}$ are geodesically convex, hence so their intersections. $\boxtimes$

Elliptic differential operators on vector bundles

$E \xrightarrow{\pi} M$ smooth vectorbundle

$$C^\infty(M,E) := \{ u : M \to E \mid u \text{ smooth}, \ \pi \circ u = id_M \}$$

Def: A linear 2nd order differential operator on E is a linear map $L : C^\infty(M,E) \to C^\infty(M,E)$, s.t. locally

$$Lu = a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + cu$$

Here, $E|_U \cong U \times \mathbb{R}^r$ and $x : U \to V \subset \mathbb{R}^n$,

so $\mathbb{R}^n \supset V \xrightarrow{u} \mathbb{R}^r$ and $a^{ij}(x), b^i(x), c(x) : \mathbb{R}^r \to \mathbb{R}^r$ linear

Def: L is elliptic $:\Leftrightarrow a^{ij}(x)\xi_i\xi_j : E_x \to E_x$ is invertible for all $0 \neq \xi \in T_x^* M$, $\forall x \in M$.

Note: $a^{ij}(x)\xi_i\xi_j$ is independent of choice of coordinates.

Ex (M,g) closed, connected Riemannian mfd.

On functions (i.e. sections of the trivial line bundle) Δ is defined by

$$-\int_M \Delta u \, v \, dV = \int_M \langle du, dv \rangle \, dV$$

In local coordinates:

$$dV = \sqrt{\det g_{ij}}\, dx^1 \wedge \dots \wedge dx^n \quad \text{where } g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

$$\langle du, dv \rangle = g^{ij}\partial_i u\, \partial_j v \quad \text{where } g^{ij} g_{jk} = \delta^i_k$$

$$\Rightarrow \Delta u = \frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}\, g^{ij}\partial_j u\right) = g^{ij}\partial_i\partial_j u + \text{l.o.t.}$$

So $\forall \xi \neq 0 \quad g^{ij}\xi_i\xi_j = |\xi|^2 : \mathbb{R} \to \mathbb{R}$ invertible,
i.e. Δ indeed elliptic.

Also: $\Delta u = 0 \Rightarrow \int_M |du|^2 = 0 \Rightarrow u = \text{const}$, i.e. $\ker(\Delta) = \mathbb{R}$

$\int_M \Delta u \cdot v = \int_M u\, \Delta v$ (formally selfadjoint)

$\Rightarrow \operatorname{coker}(\Delta) = \mathbb{R}$, namely $\Delta u = f$ has solution $\Leftrightarrow \int_M f = 0$.

Goal: Generalize $C^\infty(M, \mathbb{R}) = \operatorname{im}(\Delta) \oplus \underbrace{\ker(\Delta)}_{\text{finite dim}}$ (in fact 1-dim in our Ex)
to elliptic operators on vector bundles.

Let $E \xrightarrow{\pi} M$ be a smooth vectorbundle with inner product $\langle , \rangle$

eg on TM have $\langle X, Y \rangle = g_{ij} X^i Y^j$
on T^*M have $\langle \xi, \xi \rangle = g^{ij}\xi_i\xi_j$

Consider $\langle u, v \rangle_{L^2} := \int_M \langle u, v \rangle\, dV$

Def: L is formally selfadjoint if $\langle Lu, v \rangle_{L^2} = \langle u, Lv \rangle_{L^2}$

$\forall u, v \in C^\infty(M, E)$.

Let $L^2(M,E)$ be the completion of $C^\infty(M,E)$ in the norm $\|u\|_{L^2} = \sqrt{\langle u,u\rangle_{L^2}}$

<u>Note:</u> Fixing finite atlas, local trivializations $E|_{U_\alpha} \cong U_\alpha \times \mathbb{R}^r$ and subordinate partition of unity ρ_α have

$$u = \sum_\alpha \rho_\alpha u$$

represented by $\mathbb{R}^n \supset V_\alpha \xrightarrow{u_\alpha} \mathbb{R}^r$

So $$\|u\|^2_{L^2} \sim \sum_\alpha \int_{V_\alpha} u_\alpha^2$$

(equivalent norm)

To describe functions whose first k derivatives are in L^2, set

$$\|u\|^2_{W^{k,2}} := \sum_\alpha \int_{V_\alpha} \sum_{|I|\le k} |D^I u_\alpha|^2$$

and let $W^{k,2}(M,E)$ be the completion of $C^\infty(M,E)$ in this norm.

<u>Lemma</u> (i) For $k > \ell + \frac{n}{2}$ we have

$$W^{k,2}(M,E) \hookrightarrow C^\ell(M,E) \qquad \text{(Sobolev embedding)}$$

(ii) If L is elliptic, then

$$\|u\|_{W^{k+2,2}} \le C_k \cdot \left(\|Lu\|_{W^{k,2}} + \|u\|_{L^2}\right)$$

(Garding's inequality)

Proof: Suffices to check these for $u \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^r)$.

(i) Denoting by $\hat{u}$ the Fourier transform, we have

$$u(x) = \int_{\mathbb{R}^n} e^{ix\xi}\, \hat{u}(\xi)\, d\xi$$

$$\Rightarrow D^I u = \int_{\mathbb{R}^n} e^{ix\xi} (i\xi)^I\, \hat{u}(\xi)\, d\xi$$

$$\text{so } \|u\|^2_{W^{k,2}} \sim \int_{\mathbb{R}^n} (1+|\xi|^2)^k\, |\hat{u}(\xi)|^2\, d\xi .$$

Thus, for $|I| \le \ell < k - \frac{n}{2}$ we can estimate:

$$|D^I u| \le \int_{\mathbb{R}^n} |\xi|^I\, |\hat{u}(\xi)|\, d\xi$$

$$\le \underbrace{\left(\int_{\mathbb{R}^n} \frac{|\xi|^{2I}}{(1+|\xi|^2)^k}\, d\xi \right)^{1/2}}_{\le C \text{ since } 2k-2\ell > n} \cdot \underbrace{\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^k\, |\hat{u}(\xi)|^2 d\xi \right)^{1/2}}_{\sim \|u\|_{W^{k,2}}}$$

Namely, $\|u\|_{C^\ell} \le C_{k,\ell} \|u\|_{W^{k,2}}$.

(ii) $$\lambda \int |D^2 u|^2 \underset{\uparrow}{\le} \int |a \cdot D^2 u|^2 = \int |Lu - bDu - cu|^2$$
($\lambda > 0$ from ellipticity) $$\le C\left(\int (Lu)^2 + |Du|^2 + u^2 \right)$$

But $C\int |Du|^2 \le \frac{\lambda}{2} \int |D^2 u|^2 + C\int u^2 \Rightarrow$ Garding's for $k=0$.

For general k, take k derivatives of Lu, then argue similarly. $\boxed{\,}$

Prop Suppose $L : C^\infty(M,E) \to C^\infty(M,E)$ is elliptic. Then:

(i) $\ker L$ is finite-dimensional

(ii) If L is formally selfadjoint, then we have the L^2-orthogonal decomposition $C^\infty(M,E) = \operatorname{im} L \oplus \ker L$.

Proof (i) $\left.\begin{matrix} Lu = 0 \\ \|u\|_{L^2} \le 1 \end{matrix}\right\} \underset{\substack{\text{Garding} \\ + \\ \text{Sobolev}}}{\Longrightarrow} u \in C^\infty$ with $\|u\|_{C^\ell} \le C_\ell \ \forall \ell$.

Now, if $\dim \ker L = \infty$, then $D := \{u \in \ker L : \|u\|_{L^2} \le 1\} \subset L^2$ is noncompact

However, $u_i \in D \Rightarrow \|u_i\|_{C^1} \le C \underset{\text{Arzela-Ascoli}}{\Longrightarrow} u_{i'} \to u$ in C^0, in particular in L^2 ↯.

(ii) If $u \in \ker L$, then $\langle u, Lv\rangle = 0$, so $\ker L \perp_{L^2} \operatorname{im} L$.

We will show: $L^2(M,E) = L(W^{2,2}(M,E)) \oplus \ker L$ $(*)$

(by Garding this implies $W^{k,2}(M,E) = L(W^{k+2,2}(M,E)) \oplus \ker L$ and thus the assertion)

To prove $(*)$ is suffices[*)] to show that $L : W^{2,2} \to L^2$ has closed range.

So let $Lu_i \to f$ in L^2, $u_i \in L^2$, wlog $u_i \perp \ker L$.

Suppose towards a contradiction $\|u_i\|_{L^2} \to \infty$. Set $v_i := \frac{u_i}{\|u_i\|_{L^2}}$.

Then $Lv_i \to 0$ in L^2, $\|v_i\|_{L^2} = 1 \underset{\text{Garding}}{\Longrightarrow} \|v_i\|_{W^{2,2}} \le C \underset{\text{Rellich-Kondrachov}}{\Longrightarrow} v_{i'} \to v$ in L^2

But $v \in \ker L \cap (\ker L)^\perp$ so $v = 0$ ↯ $\|v_i\|_{L^2} = 1$.

This shows $\|u_i\|_{L^2} \le C$. Garding $\Rightarrow \|u_i\|_{W^{2,2}} \le C \underset{\text{Rellich-Kondrachov}}{\Longrightarrow} u_i \to u$ in L^2

$$\begin{matrix} \langle Lu_i, v\rangle = \langle u_i, Lv\rangle \\ \downarrow \qquad\quad \downarrow \\ \langle f, v\rangle = \langle u, Lv\rangle \end{matrix}$$ so $Lu = f$ i.e. L has closed range. ☒

*) Since for H Hilbert, $V \perp W \subset H$ closed subspaces: $V^\perp \cap W^\perp = \{0\} \Rightarrow H = V \oplus W$.

Hodge decomposition & Poincare duality

Motivation: For vectorfields on $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ one has the Helmholtz decomposition: $X = \underbrace{\operatorname{grad}\varphi}_{\text{curl free}} + \underbrace{\operatorname{curl}\vec{A}}_{\text{div free}} + \underbrace{X_0}_{\substack{\text{constant} \\ (\text{curl \& div free})}}$

Consequently,

$$H^1(T^3) \cong \frac{\{X \mid \operatorname{curl} X = 0\}}{\{X \mid X = \operatorname{grad}\varphi\}} \cong \{X_0 \mid X_0 \text{ const.}\} \cong \frac{\{X \mid \operatorname{div} X = 0\}}{\{X \mid X = \operatorname{curl}\vec{A}\}} \cong H^2(T^3)$$

Goal: Generalize this to mfds & arbitrary dim.

Setup: M oriented closed smooth n-dim mfd.

Equip M with some Riemannian metric g.

$e_1, \dots, e_n \in T_xM$ ONB, i.e. $g_x(e_i, e_j) = \delta_{ij}$.

$e^1, \dots, e^n \in T_x^*M$ dual basis, i.e. $e^i(e_j) = \delta^i_j$.

Note: $\Lambda^k T_x^*M$ has natural inner product $\langle\, , \rangle$, s.t. $\{e^{i_1} \wedge \dots \wedge e^{i_k}\}_{1 \le i_1 < \dots < i_k \le n}$ is ONB

Def: The Hodge star operator $* : \Lambda^k T_x^*M \to \Lambda^{n-k} T_x^*M$ is defined by $\omega \wedge *\eta = \langle \omega, \eta \rangle dV$ for all $\omega, \eta \in \Lambda^k T_x^*M$.

Note: Equivalently, $*(e^{i_1} \wedge \dots \wedge e^{i_k}) = \sigma e^{j_1} \wedge \dots \wedge e^{j_{n-k}}$, where $(j_1, \dots, j_{n-k})$ is the complementary multiindex of $(i_1, \dots, i_k)$, and $\sigma = \pm 1$ if $(1, \dots, n) \mapsto (i_1, \dots, i_k, j_1, \dots, j_{n-k})$ is even/odd.

Ex On T^3 we have $dV = dx\wedge dy \wedge dz$, so

$*1 = dx\wedge dy\wedge dz$	$*(dx\wedge dy\wedge dz) = 1$
$*dx = dy\wedge dz$	$*(dy\wedge dz) = dx$
$*dy = -dx\wedge dz$	$*(dx\wedge dz) = -dy$
$*dz = dx\wedge dy$	$*(dx\wedge dy) = dz$

Note: In general, $* : \Lambda^k T_x^* M \to \Lambda^{n-k} T_x^* M$ is an isometry with $** = (-1)^{k(n-1)}$.

Now consider $L^2(M, \Lambda^k T^*M)$ with the L^2-inner product

$$\langle\langle \omega, \eta \rangle\rangle := \int_M \langle \omega, \eta \rangle \, dV.$$

Def: The coderivative $d^* : C^\infty(M, \Lambda^{k+1} T^*M) \to C^\infty(M, \Lambda^k T^*M)$ is defined by $d^* := (-1)^{nk+1} * d *$

Prop: d^* is the formal adjoint of d, namely

$$\langle\langle d\omega, \eta \rangle\rangle = \langle\langle \omega, d^*\eta \rangle\rangle \text{ for all } \omega \in C^\infty(M, \Lambda^k T^*M), \ \eta \in C^\infty(M, \Lambda^{k+1} T^*M)$$

Proof: $\langle\langle d\omega, \eta \rangle\rangle = \int_M \langle d\omega, \eta \rangle dV = \int_M d\omega \wedge *\eta \underset{\text{Stokes}}{=} (-1)^{k+1} \int_M \omega \wedge d*\eta$

But now $d*\eta = (-1)^{k(n-1)} ** d * \eta$,

so $\langle\langle d\omega, \eta \rangle\rangle = (-1)^{kn+1} \langle\langle \omega, *d*\eta \rangle\rangle$ ◻

Def: $-\Delta := dd^* + d^*d$ is called the Hodge Laplacian

Note: Δ is formally selfadjoint, i.e. $\langle\langle \Delta\omega, \eta\rangle\rangle = \langle\langle \omega, \Delta\eta\rangle\rangle$
for all $\omega, \eta \in C^\infty(M, \Lambda^k T^*M)$

Ex On $T^n = \mathbb{R}^n/\mathbb{Z}^n$ with the flat metric $g_{ij} = \delta_{ij}$ we have

$$\langle\langle d^*\omega, \eta\rangle\rangle = \langle\langle \omega, d\eta\rangle\rangle$$
$$= \int_{T^n} \sum_I \omega_I (d\eta)_I \, dV$$
$$= \int_{T^n} \sum_{I=(j,J)} \omega_I \frac{\partial \eta_J}{\partial x^j} \, dV$$
$$= -\int_{T^n} \sum_{I=(j,J)} \frac{\partial \omega_I}{\partial x^j} \eta_J \, dV$$

$\omega = \sum_I \omega_I \, dx^I$

$I = (i_1, \dots, i_k)$

$dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$\eta = \sum_J \eta_J \, dx^J \quad |J| = k-1$

$d\eta = \sum_{j,J} \frac{\partial \eta_J}{\partial x^j} dx^j \wedge dx^J$

$$\Rightarrow \quad d^*\omega = -\sum_{j,I} \frac{\partial \omega_I}{\partial x^j} \, i_{\frac{\partial}{\partial x^j}} dx^I$$

Thus
$$\begin{cases} -dd^*\omega = \sum_{I,j,k} \frac{\partial^2 \omega_I}{\partial x^j \partial x^k} dx^k \wedge \left(i_{\frac{\partial}{\partial x^j}} dx^I \right) \\ -d^*d\omega = \sum_{I,j,k} \frac{\partial^2 \omega_I}{\partial x^j \partial x^k} i_{\frac{\partial}{\partial x^j}} \left(dx^k \wedge dx^I \right) \end{cases}$$

$$\Rightarrow \quad \boxed{\Delta\omega = \sum_I \left(\sum_j \frac{\partial^2 \omega_I}{\partial x_j^2} \right) dx^I}$$

i_X derivation of degree -1

On (M,g) in terms of local ON frame still have

$$\omega = \sum_I \omega_I e^I \quad , \quad e^I = e^{i_1} \wedge \ldots \wedge e^{i_k}$$

$$d\omega = \sum_{I,j} e_j(\omega_I)\, e^j \wedge e^I + \sum_I \omega_I\, de^I$$

$$d^*\omega = -\sum_{I,j} e_j(\omega_I)\, i_{e_j} e^I + \sum_{I,K} \alpha_{I,K}\, \omega_I\, e^K$$

$$\Rightarrow \Delta\omega = \sum_{I,i,j} g^{ij} \frac{\partial^2 \omega_I}{\partial x^i \partial x^j} dx^I + \text{l.o.t.}$$

Note: Multiplication by $g^{ij}\xi_i\xi_j = |\xi|^2$ is invertible for $\xi \neq 0$, so Δ is elliptic!

Def: $\mathcal{H}^k(M) := \{ \omega \in C^\infty(M, \Lambda^k T^*M) \mid \Delta\omega = 0 \}$ is called the space of harmonic k-forms.

Note: Since $\langle\langle \omega, -\Delta\omega \rangle\rangle = \|d\omega\|^2 + \|d^*\omega\|^2$ we have

$$\Delta\omega = 0 \iff \underbrace{d\omega = 0}_{\text{closed}} \;\&\; \underbrace{d^*\omega = 0}_{\text{coclosed}}$$

Thm (Hodge) We have the L^2-orthogonal decomposition

$$C^\infty(M, \Lambda^k T^*M) = \operatorname{im}(d) \oplus \operatorname{im}(d^*) \oplus \mathcal{H}^k(M),$$

where $\operatorname{im} d := d(C^\infty(M, \Lambda^{k-1}T^*M))$, $\operatorname{im} d^* := d^*(C^\infty(M, \Lambda^{k+1}T^*M))$.

In particular, $\begin{aligned} \mathcal{H}^k(M) &\longrightarrow H^k(M) \\ \omega &\longmapsto [\omega] \end{aligned}$ is an isomorphism,

i.e. each de Rham cohomology class has a unique harmonic representative.

Proof: Clearly, $\mathcal{H}^k(M)$ is orthogonal to both $\mathrm{im}(d)$ & $\mathrm{im}(d^*)$.

Also $\langle\langle d\omega, d^*\eta\rangle\rangle \underset{d\circ d=0}{=} 0$, so $\mathrm{im}(d) \perp \mathrm{im}(d^*)$ as well.

Now, by the elliptic theory from last lecture, we have

$$C^\infty(M, \Lambda^k T^*M) = \mathrm{im}(\Delta) \oplus \underbrace{\ker(\Delta)}_{\text{finite-dim}}$$

Since $\ker(\Delta) = \mathcal{H}^k(M)$ and $\mathrm{im}(\Delta) \subseteq \mathrm{im}(d) \oplus \mathrm{im}(d^*)$ this yields

$$C^\infty(M, \Lambda^k T^*M) = \mathrm{im}(d) \oplus \mathrm{im}(d^*) \oplus \mathcal{H}^k(M).$$

In particular: $H^k(M) = \dfrac{\ker d}{\mathrm{im}\, d} = \dfrac{\mathrm{im}(d^*)^\perp}{\mathrm{im}\, d} \cong \mathcal{H}^k(M)$. $\boxtimes$

Rmk: In particular, this gives a new proof of $\dim H^k(M) < \infty$.

Cor (Poincare duality) The bilinear pairing

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R}$$
$$([\omega], [\eta]) \longmapsto \int_M \omega \wedge \eta \text{ is nondegenerate.}$$

In particular, $\dim H^k(M) = \dim H^{n-k}(M)$.

Proof Immediate, since $\mathcal{H}^k(M) \to \mathcal{H}^{n-k}(M)$, $\omega \longmapsto *\omega$ is an isomorphism. $\boxtimes$

Ex $H^k(T^n) \cong \mathbb{R}^{\binom{n}{k}} \cong H^{n-k}(T^n)$ with harmonic representatives the constant forms $\{dx^{i_1} \wedge \ldots \wedge dx^{i_k}\}_{1 \le i_1 < \ldots < i_k \le n}$ respectively their duals.

Cor The Betti numbers $b_k(M) := \dim H^k(M)$ satisfy $b_k = b_{n-k}$.

In particular, if n is odd, then the Euler characteristic $\chi(M) := \sum_k (-1)^k b_k(M)$ vanishes.

Cor (Künneth formula) $H^k(M \times N) = \bigoplus_{i+j=k} H^i(M) \otimes H^j(N)$

Proof:
$$\begin{tikzcd} & M \times N \arrow[dl, "\pi_M"'] \arrow[dr, "\pi_N"] & \\ M & & N \end{tikzcd}$$
$\pi_M^*\omega \cdot \pi_N^*\eta$ harmonic on $M \times N \Leftrightarrow \Delta_M \omega = 0$ & $\Delta_N \eta = 0$ $\boxtimes$