De Rham Cohomology

Motivation: $\vec{E} = \operatorname{grad} \varphi \Rightarrow \operatorname{curl} \vec{E} = 0$
$\vec{B} = \operatorname{curl} \vec{A} \Rightarrow \operatorname{div} \vec{B} = 0$

But the converse is not quite true, e.g.

$$\vec{E} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \text{ on } \mathbb{R}^3 \setminus \{x=y=0\}$$

$$\vec{B} = \frac{1}{x^2+y^2+z^2}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ on } \mathbb{R}^3 \setminus \{0\}$$

don't have a potential.

Goal: Capture the obstruction systematically on manifolds of arbitrary dimension.

Recall: $d \circ d = 0$.

Namely if $\omega = d\eta$, then $d\omega = 0$.

Terminology: A differential form ω is called

-) <u>closed</u> if $d\omega = 0$
-) <u>exact</u> if $\exists \eta$ st $\omega = d\eta$

So $d \circ d = 0$ becomes "exact $\Rightarrow$ closed"

<u>Def</u>: Let M be a smooth manifold. Its <u>de Rham cohomology</u> $H^*(M)$ is defined by

$$\boxed{H^k(M) := \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}}}$$

Here $\{\text{closed } k\text{-forms}\} = \{\omega \in \Omega^k(M) \mid d\omega = 0\}$

$$= \ker\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right)$$

and $\{\text{exact } k\text{-forms}\} = \{\omega \in \Omega^k(M) \mid \exists \eta : \omega = d\eta\}$

$$= \operatorname{im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right)$$

<u>Terminology</u>: The elements $[\omega] \in H^k(M)$ are called <u>cohomology classes</u>.

By definition $[\omega] = [\omega'] \iff \exists \lambda : \omega - \omega' = d\lambda$.

<u>Note</u>: $H^k(M)$ is defined as quotient of real vector spaces and thus a real vector space (in particular an abelian group)

<u>Note</u>: $H^k(M) = 0$ for $k > \dim M$.

Ex ·) $H^k(\mathbb{R}) \cong \begin{cases} \mathbb{R} & k=0 \\ 0 & \text{else} \end{cases}$

indeed $H^0(\mathbb{R}) = \{\text{constant functions on } \mathbb{R}\} \cong \mathbb{R}$ and every $\omega = a(x)dx \in \Omega^1(\mathbb{R})$ can be written as $\omega = d\left(\int_0^x a(y)dy\right)$.

·) $H^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & k=0 \\ 0 & \text{else} \end{cases}$ (next hour)

·) $H^k(S^1) \cong \begin{cases} \mathbb{R} & k=0,1 \\ 0 & \text{else} \end{cases}$ (last HW)

·) $H^k(S^n) \cong \begin{cases} \mathbb{R} & k=0,n \\ 0 & \text{else} \end{cases}$ (next week)

·) $H^k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{R} & 0 \le k \le 2n,\ k \text{ even} \\ 0 & \text{else} \end{cases}$ (next week)

·) $H^k(T^n) \cong \begin{cases} \mathbb{R}^{\binom{n}{k}} & 0 \le k \le n \\ 0 & \text{else} \end{cases}$ (next week)

·) $H^k(\Sigma_g) \cong \begin{cases} \mathbb{R} & k=0,2 \\ \mathbb{R}^{2g} & k=1 \\ 0 & \text{else} \end{cases}$ (new HW).

First basic properties

- $H^0(M) = \{f : M \to \mathbb{R} \mid df = 0\} \cong \mathbb{R}^{\pi_0(M)}$,

  where $\pi_0(M) = \{\text{connected components}\}$

  In part: M connected $\Leftrightarrow H^0(M) = \mathbb{R}$

- If $M = \coprod_i M_i$ (disjoin union), then

  $$H^*(M) \cong \prod_i H^*(M_i) \text{ induced by } \omega \mapsto (\omega|_{M_1}, \omega|_{M_2}, \dots).$$

  So can assume from now on M connected.

- $f : M \to N$ diffeo.

  pullback $f^* : \Omega^*(N) \to \Omega^*(M)$ commutes with d,
  so get well-defined linear map

  $$f^* : H^*(N) \to H^*(M)$$
  $$[\omega] \longmapsto [f^*\omega]$$

  indeed: $d\omega = 0 \Rightarrow df^*\omega = 0$

  and $f^*(\omega + d\eta) = f^*\omega + d(f^*\eta)$

  Moreover, $(f \circ g)^* = g^* \circ f^*$ & $id^* = id$, i.e.
  deRham cohomology is a (contravariant) functor
  from smooth mfds to real vector spaces.

  In part: f diffeo $\Rightarrow f^*$ vector space isomorphism.

- Similarly, $H^k(M) \times H^\ell(M) \longrightarrow H^{k+\ell}(M)$
$$([\omega],[\eta]) \longmapsto [\omega \wedge \eta]$$
is well-defined.

So $H^*(M) = \bigoplus_{k=0}^{n} H^k(M)$ is a graded algebra.

Q: Top cohomology?

Prop Let M be a smooth connected closed n-manifold.

Then (i) $H^n(M) = \mathbb{R}$ if M is orientable

(ii) $H^n(M) = 0$ if M is not orientable.

Proof: (i) already seen last time.

(ii) Consider $\tilde{M} := \{(p,o) \mid p \in M, o \text{ orientation of } T_pM\}$

Note $\tilde{M} \xrightarrow{\pi} M$ is $2:1$ & loc diffeo & $\tilde{M}$ is connected.

Also $\alpha: \tilde{M} \to \tilde{M}$, $(p,o) \mapsto (p,-o)$ is an orientation reversing diffeo satisfying $\pi \circ \alpha = \pi$, $\alpha^2 = \mathrm{id}$.

Claim: $\pi^*: H^n(M) \to H^n(\tilde{M})$ is injective.

Proof of the claim:

Suppose $\omega \in \Omega^n(M)$ is such that

$\pi^*[\omega] = 0 \in H^n(\tilde{M})$, i.e. $\exists \tilde{\eta} \in \Omega^{n-1}(\tilde{M})$

st $\pi^*\omega = d\tilde{\eta}$

Can assume $\alpha^*\tilde{\eta} = \tilde{\eta}$

(if not, replace $\tilde{\eta}$ by $\frac{1}{2}(\tilde{\eta} + \alpha^*\tilde{\eta})$)

So $\exists \eta \in \Omega^{n-1}(M)$ with $\pi^*\eta = \tilde{\eta}$.

Thus, $\omega = d\eta$, i.e. $[\omega] = 0 \in H^n(M)$. $\square$

Now, given $\omega \in \Omega^n(M)$ consider $\tilde{\omega} := \pi^*\omega$

Then, $\alpha^*\tilde{\omega} = \tilde{\omega}$, so

$$\int_{\tilde{M}} \tilde{\omega} = \int_{\tilde{M}} \alpha^*\tilde{\omega} \underset{\substack{\uparrow \\ \alpha \text{ orient.} \\ \text{rev.}}}{=} -\int_{\tilde{M}} \tilde{\omega}, \text{ i.e. } \int_{\tilde{M}} \tilde{\omega} = 0$$

$$\text{part (i)} \Rightarrow [\tilde{\omega}] = 0 \in H^n(\tilde{M}) \underset{\text{Claim}}{\Longrightarrow} H^n(M) = 0 \quad \square$$

Cor $f: M^n \to N^n$ smooth map between oriented connected closed smooth mfds

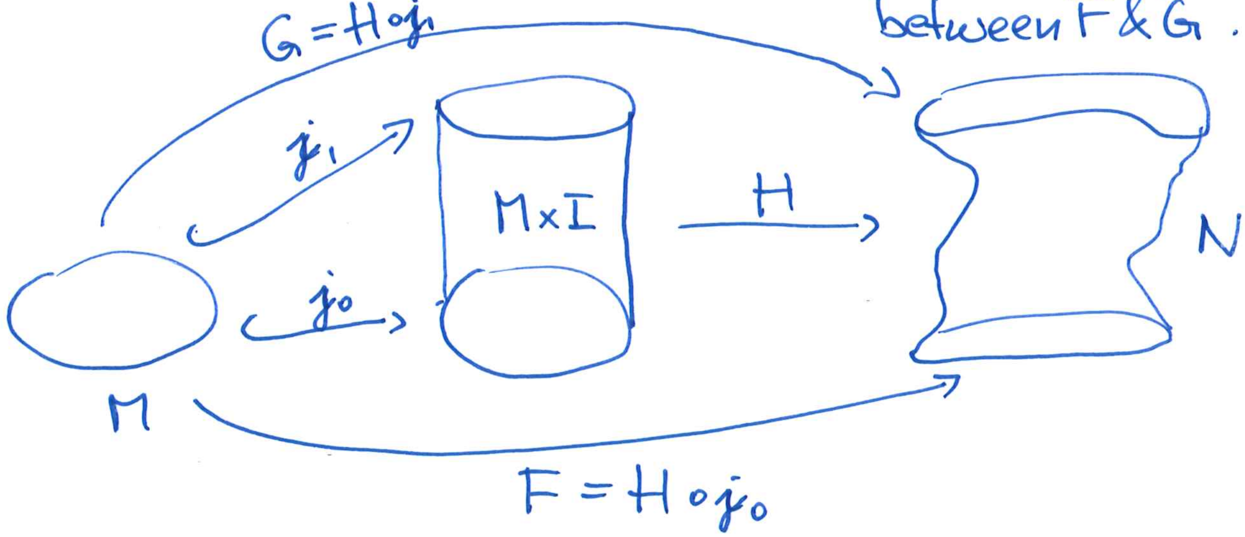
$\Rightarrow f^*: H^n(N^n) \underset{\uparrow}{\cong} \mathbb{R} \longrightarrow \mathbb{R} \underset{\uparrow}{\cong} H^n(M^n)$ is multiplication by $\deg(f)$.

(canonical iso via integration)

Thm (homotopy invariance)

If $F, G : M \to N$ are homotopic, then the induced maps $F^*, G^* : H^k(N) \to H^k(M)$ are equal.

Proof: Let $H : M \times I \to N$ be a smooth homotopy between F & G.



Now, $F^* = j_0^* \circ H^*$ & $G^* = j_1^* \circ H^*$, so it is clearly enough to show that $j_0^*, j_1^* : H^k(M \times I) \to H^k(M)$ are equal.

Let $[\omega] \in H^k(M \times I)$. Then:

$$j_1^* \omega - j_0^* \omega \underset{\uparrow}{=} dh\omega \implies [j_1^* \omega] = [j_0^* \omega]$$

$d\omega = 0$ & Lemma below

$\square$

Lemma (existence of homotopy operator)

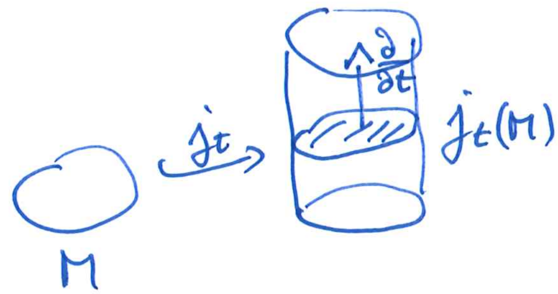
$$\exists\, h : \Omega^*(M \times I) \to \Omega^{*-1}(M)$$

s.t. $\boxed{j_1^* - j_0^* = hd + dh}$

Proof: For $\omega \in \Omega^k(M\times I)$ define $h\omega \in \Omega^{k-1}(M)$ by

$$h\omega := \int_0^1 j_t^*\, i_{\frac{\partial}{\partial t}} \omega \; dt,$$

where $j_t : M \to M \times I$
$\qquad x \mapsto (x,t)$



$$\Rightarrow \begin{cases} dh\omega = \int_0^1 j_t^*\, d\, i_{\frac{\partial}{\partial t}} \omega \; dt \\ hd\omega = \int_0^1 j_t^*\, i_{\frac{\partial}{\partial t}} d\omega \; dt \end{cases}$$

Cartan's magic formula $\Rightarrow$ $dh\omega + hd\omega = \int_0^1 j_t^* \mathcal{L}_{\frac{\partial}{\partial t}} \omega \; dt$

Finally, for $\frac{\partial}{\partial t}$ the flow φ_t is simply $\varphi_t(x,s) = (x, s+t)$.

So $j_t = \varphi_t \circ j_0$ and $\frac{d}{dt} \varphi_t^* \omega = \varphi_t^* \mathcal{L}_{\frac{\partial}{\partial t}} \omega$

yields $j_t^* \mathcal{L}_{\frac{\partial}{\partial t}} \omega = \frac{d}{dt} j_t^* \omega \underset{\text{FTC}}{\Longrightarrow} dh\omega + hd\omega = j_1^*\omega - j_0^*\omega$ ∎

Cor (homotopy invariance)

If M, N are homotopy equivalent smooth mfds, then $H^*(M) \cong H^*(N)$.

Proof $M \underset{G}{\overset{F}{\rightleftarrows}} N$ st $G \circ F \sim id_M$, $F \circ G \sim id_N$

homotopic

So by the thm we get $F^* \circ G^* = id$, $G^* \circ F^* = id$, i.e. $F^* : H^k(N) \to H^k(M)$ is an isomorphism with inverse G^*. $\boxtimes$

Cor (top invariance)

If M, N are homeomorphic smooth mfds, then $H^*(M) \cong H^*(N)$.

Rmk In part, H^* does not depend on smooth structure.

Cor (Poincare lemma)

If $U \subset M$ is contractible, then $H^k(U) = 0 \ \forall k \geq 1$

i.e. $\forall k \geq 1 \ \forall \omega \in \Omega^k(U)$ with $d\omega = 0 \ \exists \lambda \in \Omega^{k-1}(U) : \omega = d\lambda$.

Rmk In part, holds for $U \subset \mathbb{R}^n$ (or $\mathbb{H}^n$) starshaped.

Cor (invariance of dimension) $\underset{\text{homeomorphic}}{\mathbb{R}^n \cong \mathbb{R}^m} \Rightarrow n = m$

Proof $H^k(\mathbb{R}^n \setminus p) \cong H^k(S^{n-1})$. $\boxtimes$ **Rmk** In part, dim of top. mfd. well-defined.

Q: First cohomology vs fundamental group?

M connected smooth mfd.

Define $\Phi : H^1(M) \longrightarrow \mathrm{Hom}(\pi_1(M,p), \mathbb{R})$

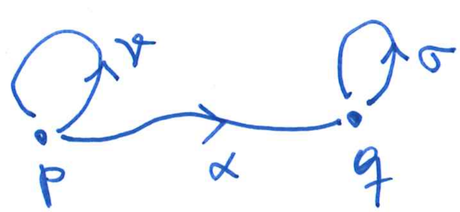
$$\Phi[\omega][\gamma] := \int_\gamma \omega \quad (\gamma \text{ smooth representative})$$

well-defined: $\cdot)$ $\int_{\gamma * \tilde{\gamma}} \omega = \int_\gamma \omega + \int_{\tilde{\gamma}} \omega$.

$\cdot)$ $\omega' = \omega + df \Rightarrow \int_\gamma \omega' - \int_\gamma \omega = \int_\gamma df = 0$

$\cdot)$ $[\gamma] = [\gamma']$ in $\pi_1 \Rightarrow \int_\gamma \omega = \int_{\gamma'} \omega$ by Stokes $\boxtimes$

Now, suppose $\Phi[\omega] = 0$, i.e. $\int_\gamma \omega$ $\forall \gamma$ loop at p.



$$\Rightarrow 0 = \int_{\alpha^{-1}\sigma\alpha} \omega = \int_\alpha \omega + \int_\sigma \omega - \int_\alpha \omega \Rightarrow \int_\sigma \omega = 0$$

$\forall \sigma$ loop at q

$\Rightarrow \omega = df$, i.e. Φ is injetive.

In part: $\boxed{\pi_1 = 0 \Rightarrow H^1 = 0}$

Rmk: With more work, can show Φ is isomorphism.

Algebra of differential complexes

recall: The space of differential forms $\Omega^*(M) = \bigoplus_{k=0}^{n} \Omega^k(M)$ comes equipped with differential

$$\cdots \to \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to \cdots$$

which satisfies $d \circ d = 0$. This allowed us to introduce

$$H^k(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}.$$

Moreover, for any $f : M \to N$ smooth, pull back yields

$$\begin{array}{ccccccccc}
\cdots \to & \Omega^{k-1}(N) & \xrightarrow{d} & \Omega^k(N) & \xrightarrow{d} & \Omega^{k+1}(N) & \to \cdots \\
& \downarrow f^* & & \downarrow f^* & & \downarrow f^* & \\
\cdots \to & \Omega^{k-1}(M) & \xrightarrow{d} & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \to \cdots
\end{array}$$

which commutes, namely $d f^* = f^* d$.

So, this (functorially) induces maps

$$f^* : H^k(N) \longrightarrow H^k(M)$$

in de Rham cohomology.

goal: capture algebraic setting systematically, and derive some general consequences.

Defs .) A differential complex is given by

$$A^* = \bigoplus_k A^k \quad \text{abelian groups (or vectorspaces)}$$

and homomorphisms (or linear maps)

$$\ldots \to A^{k-1} \xrightarrow{d} A^k \xrightarrow{d} A^{k+1} \to \ldots$$

such that $d \circ d = 0$.

.) The cohomology groups of A^* are defined by

$$H^k(A^*) := \frac{\ker(d : A^k \to A^{k+1})}{\mathrm{im}(d : A^{k-1} \to A^k)} .$$

.) A chain map $f : A^* \to B^*$ between differential complexes is a collection of homomorphisms $f_k : A^k \to B^k$, such that $d_B f_k = f_{k+1} d_A$, namely

$$\begin{array}{ccccccccc} \ldots \to & A^{k-1} & \xrightarrow{d} & A^k & \xrightarrow{d} & A^{k+1} & \to \ldots \\ & \downarrow f_{k-1} & & \downarrow f_k & & \downarrow f_{k+1} & \\ \ldots \to & B^{k-1} & \xrightarrow{d} & B^k & \xrightarrow{d} & B^{k+1} & \to \ldots \end{array} \quad \text{commutes.}$$

Note: Any chain map $f : A^* \to B^*$ induces maps

$$H^k(A^*) \to H^k(B^*), \quad [a] \mapsto [f_k a]$$

in cohomology.

Indeed: $d f_k a = f_{k+1} da = 0$ and $f_k(a + d\alpha) = f_k a + d f_k \alpha$. $\square$

Exact sequences

Def: A sequence $\cdots \to A^{k-1} \xrightarrow{f_{k-1}} A^k \xrightarrow{f_k} A^{k+1} \to \cdots$

is called <u>exact</u> if $\ker(f_k) = \mathrm{im}(f_{k-1})\ \forall k$.

In particular, an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a <u>short exact sequence</u>.

(concretely: f injective, $\mathrm{im} f = \ker g$, g surjective)

Note: Having a short exact sequence tells us a great deal (but not quite everything) about the group in the middle.

Ex: If $\mathbb{Z}_2 \to G \to \mathbb{Z}_2$ is short exact,

then ~~~~ $|G| = 4$ and either $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ or $G = \mathbb{Z}_4$

Compare with: If $\mathbb{R} \to E \to S^1$ is a line bundle,

then $\dim E = 2$ and either $E = \mathbb{R} \times S^1$ or

$E =$ Möbius band.

Zig-Zag Lemma

If $0 \to A^* \to B^* \to C^* \to 0$ is a short exact sequence of differential complexes, then there exist natural homomorphisms $\delta : H^k(C^*) \to H^{k+1}(A^*)$, st

$$\begin{tikzcd} \cdots \arrow[r] & H^k(B^*) \arrow[r] & H^k(C^*) \arrow[dll, bend left] \\ H^{k+1}(A^*) \arrow[r] & H^{k+1}(B^*) \arrow[r] & \cdots \end{tikzcd}$$

is a long exact sequence.

Proof (sketch). By assumption we have a commutative diagram, where the rows are exact.

$$\begin{tikzcd} 0 \arrow[r] & A^{k+2} \arrow[r] & B^{k+2} \arrow[r] & C^{k+2} \arrow[r] & 0 \\ 0 \arrow[r] & A^{k+1} \arrow[r, "i_{k+1}"] \arrow[u] & B^{k+1} \arrow[r, "j_{k+1}"] \arrow[u] & C^{k+1} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & A^k \arrow[r, "i_k"] \arrow[u] & B^k \arrow[r, "j_k"] \arrow[u] & C^k \arrow[r] \arrow[u] & 0 \\ & \vdots & \vdots & \vdots & \end{tikzcd}$$

To construct δ, let $[x] \in H^k(C^*)$ be represented by $x \in C^k$.
Since j_k is surjective $\exists y \in B^k : x = j_k(y)$.
Now, $j_{k+1}(dy) = d j_k(y) = dx = 0$. So $dy \in \ker(j_{k+1}) = \mathrm{im}(i_{k+1})$, namely $\exists! z \in A^{k+1} : dy = i_{k+1}(z)$. Note that $dz = 0$.
Define $\delta([x]) := [z]$. Via "diagram-chasing" like above one checks that this is well-defined, natural, and that the long sequence is exact. $\square$

Ex (relative cohomology)

If $S \subset M$ is a smooth submanifold, then

$$\begin{aligned} \Omega^*(M) &\xrightarrow{r_S} \Omega^*(S) \\ \omega &\longmapsto \omega|_S \end{aligned}$$ is surjective

and d restricts to a differential on $\Omega^*(M,S) := \ker(r_S)$.

Define the relative de Rham cohomology groups by $H^*(M,S) :=$ cohomology of the complex $\Omega^*(M,S)$.

Now, by the zig-zag lemma, the short exact sequence $0 \to \Omega^*(M,S) \to \Omega^*(M) \to \Omega^*(S) \to 0$ gives rise to a long exact sequence

$$\begin{tikzcd} & & H^{k+1}(M,S) \arrow[r] & H^{k+1}(M) \arrow[r] & \cdots \\ \cdots \arrow[r] & H^k(M) \arrow[r] & H^k(S) \arrow[ur, "\delta"] & & \end{tikzcd}$$

in cohomology, known as the long exact sequence of the pair (M,S).

Intuition: $S \hookrightarrow M \to M/S$ "short exact sequence in topology"
gives $H^*(S) \leftarrow H^*(M) \leftarrow H^*(M,S)$ "long exact sequence in algebra"