Differential Topology (MAT 1300)

course website: www.math.toronto.edu/roberth/difftopo.html

Manifolds locally look like $\mathbb{R}^n$, globally more complicated

1 dim mfds (curves): 

2 dim mfds (surfaces): 

<u>Def</u>: A n-dimensional <u>topological manifold</u> is a Hausdorff & 2^{nd} countable topological space that is locally homeomorphic to $\mathbb{R}^n$.

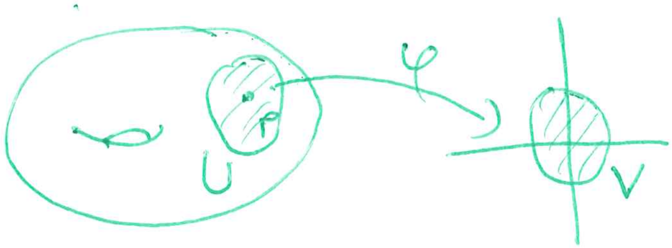
recall: M topological space

→ Hausdorff $\forall p \neq q$ $\exists$ disjoint open neighborhoods

→ 2^{nd} countable $\exists$ countable basis for topology of M

(e.g. $\{B_r(p)\}_{\substack{r \in \mathbb{Q}_+ \\ p \in \mathbb{Q}^n}}$ countable basis for topo of $\mathbb{R}^n$)

→ locally homeomorphic to $\mathbb{R}^n$



$\forall p \in M$ $\exists$ open nbd U & homeomorphism $\varphi : U \to V \subset \mathbb{R}^n$ (V open)

Nonexamples :

·) long line "$(0, \omega_1)$" (not 2^{nd} countable)

·) line with 2 origins $\mathbb{R} \sqcup \mathbb{R} / \sim$ (not Hausdorff)

glue for $x \neq 0$

Terminology

$\cdot$) (U, φ) is called a <u>coordinate chart</u>

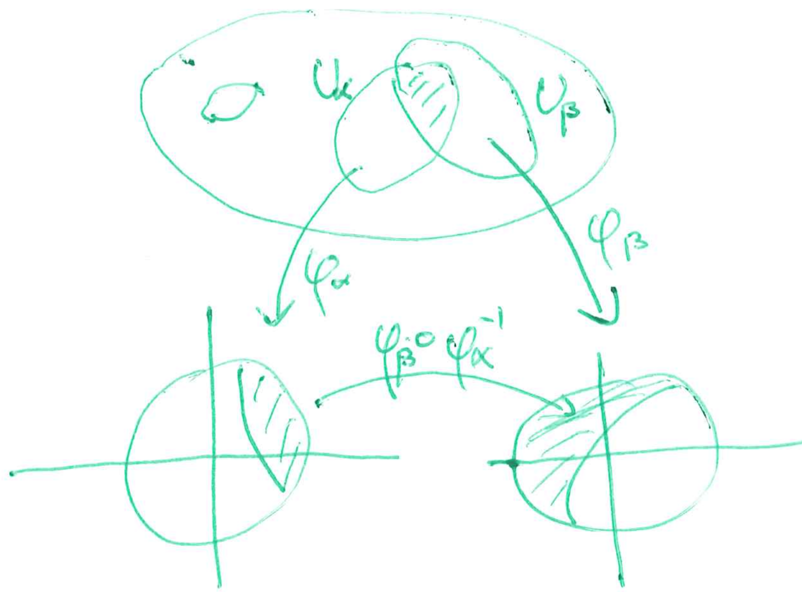
$\varphi(q) = (x^1(q), \dots, x^n(q))$ "<u>local coordinates</u>"

can arrange $\varphi(p) = 0$ "<u>chart centered at p</u>"

$\cdot$) $\{(U, \varphi) \mid (U, \varphi) \text{ coord. chart}\}$

is called <u>maximal atlas</u> of M

$\cdot$)

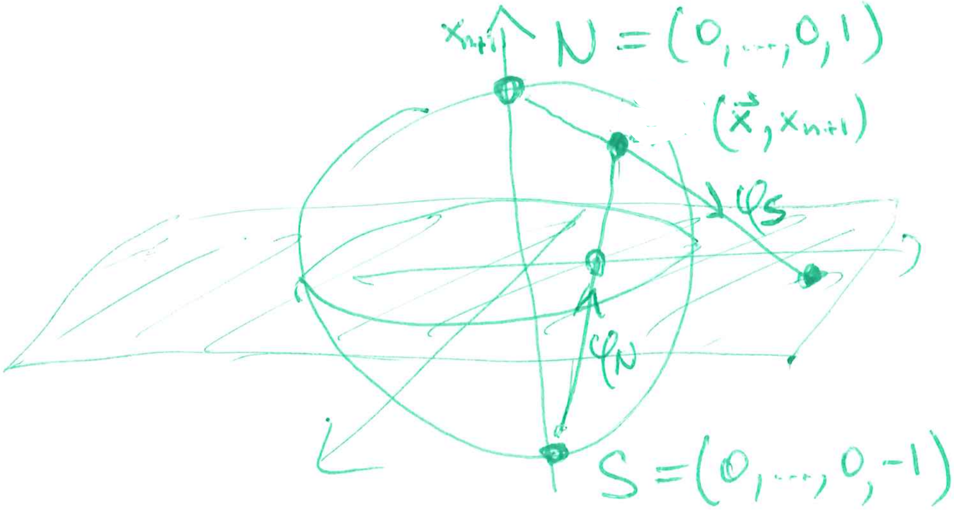


$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

are called <u>transition functions</u>

Examples & constructions

$$S^n = \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum x_i^2 = 1 \}$$



$$S^n = U_N \cup U_S$$

$$U_N = S^n \setminus \{S\}$$
$$U_S = S^n \setminus \{N\}$$

$$\varphi_N : U_N \longrightarrow \mathbb{R}^n, \quad (\vec{x}, x_{n+1}) \longmapsto \frac{\vec{x}}{1 + x_{n+1}}$$
$$\varphi_S : U_S \longrightarrow \mathbb{R}^n, \quad (\vec{x}, x_{n+1}) \longmapsto \frac{\vec{x}}{1 - x_{n+1}}$$

Note: M, N manifolds $\Rightarrow$ $M \times N$ also a manifold

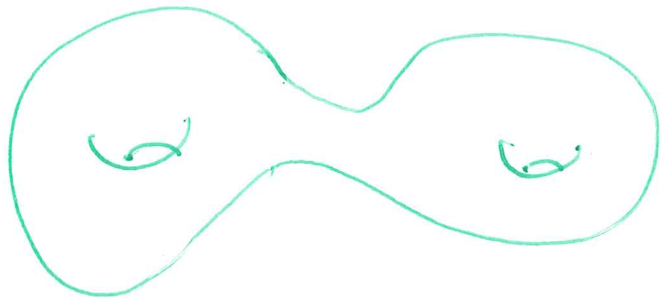
$\left(\varphi : U \to \varphi(U) \subset \mathbb{R}^n, \ \psi : V \to \psi(V) \subset \mathbb{R}^m \rightsquigarrow \varphi \times \psi : U \times V \to \varphi(U) \times \psi(V) \subset \mathbb{R}^{n+m} \right)$

e.g. $T^n = \underbrace{S^1 \times \dots \times S^1}_{n \text{ copies}}$

n-dim torus

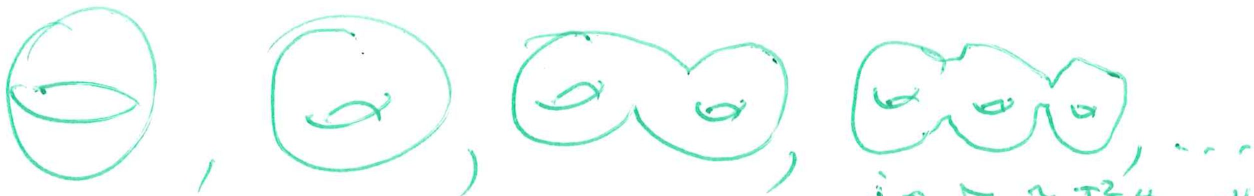
Connected sum $M \# N$

$=$



(homework)

Fact: Any orientable compact 2dim mfd M is one of:

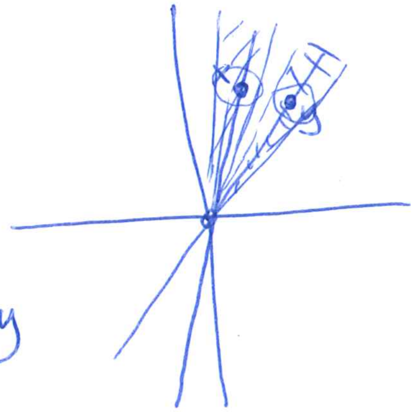


i.e. $M \cong \underbrace{T^2 \# \dots \# T^2}_{g \text{ times}}$

Here, two topological mfds $M, \tilde{M}$ are "the same" if

$\exists\, \varphi : M \longrightarrow \tilde{M}$ homeomorphism.

projective space



$$\mathbb{R}P^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \sim$$

$$x \sim y :\Leftrightarrow \exists \lambda \neq 0 : x = \lambda y$$

Note: $\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}P^n$
$\qquad x \longmapsto [x]$

is an open map.

Indeed: $U \subset \mathbb{R}^{n+1} \setminus \{0\}$ open $\Rightarrow \pi^{-1}(\pi(U)) = \bigcup_{\lambda \neq 0} \underbrace{\lambda \cdot U}_{\text{open.}} \Rightarrow \pi(U) \subset \mathbb{R}P^n$ open

def. of quotient topo

$\boxtimes$

$\Rightarrow$ Hausdorff & 2nd countable.

Charts: $\tilde{U}_i = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \neq 0\}$

$$U_i = \pi(U_i)$$

$$\varphi_i : U_i \longrightarrow \mathbb{R}^n, \ [(x_0, \dots, x_n)] \longmapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

$$\varphi_i^{-1}(y_1, \dots, y_n) = [(y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n)]$$

Exer: $\mathbb{C}P^N$ | often write $[x_0 : \dots : x_n]$

level set of smooth functions / solving equations

$$f : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}, \quad x \longmapsto \|x\|^2 = \sum x_i^2$$

$$S^n = f^{-1}(\{1\})$$

more generally $f : \mathbb{R}^{n+1} \to \mathbb{R}$ smooth

$\Longrightarrow$ $M := f^{-1}(\{c\})$ is a manifold,
provided $\nabla f(x) \neq 0 \quad \forall x \in f^{-1}(\{c\})$.

implicit function theorem.

Ex $M = \{ [x_0 : x_1 : x_2 : x_3] \in \mathbb{CP}^3 \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \}$

"K3 surface" $\dim_{\mathbb{C}} M = 2$

$\dim_{\mathbb{R}} M = 4$.

Matrix groups / Lie groups

- $Mat(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$
- $GL(n, \mathbb{R}) = \{ A \in Mat(n, \mathbb{R}) \mid \det A \neq 0 \}$

  $GL(n, \mathbb{R}) \subset Mat(n, \mathbb{R})$ open subset, hence a manifold.

  (in general: open subset of mfd is a mfd).

- $SO(n) = \{ A \in GL(n, \mathbb{R}) \mid \underset{\det A = 1}{A^T A = I} \}$

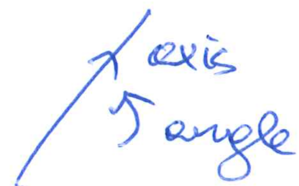
  is a manifold by implicit function thm (Exer)

- $SU(n) = \{ A \in GL(n, \mathbb{C}) : \quad A^\dagger A = I, \det A = 1 \}$
- etc

Q: Topology in low dimensions?

$\cdot)$ $SO(2) = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \right\} \cong S^1$

$\cdot)$ $SO(3)$ 3-dim



$$\begin{array}{c} SO(3) \\ \cong \\ B^3_\pi / \sim \;\cong \mathbb{RP}^3 \end{array}$$

$\uparrow$ identify antipodal points on ∂B

$\cdot)$ $SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix} \,\middle|\, |\alpha|^2 + |\beta|^2 = 1 \right\} \cong S^3$

Indeed $SU(2) \to SO(3)$ double cover.

Most manifolds come with some extra structure

	course this year
→ smooth structure	my course
→ Riemannian metric	Rotman
→ symplectic/contact form	Murphy
→ complex structure	Litt, Collins
→ positive scalar curvature metric	Liokumovich
→ Lorentzian metric	McCann

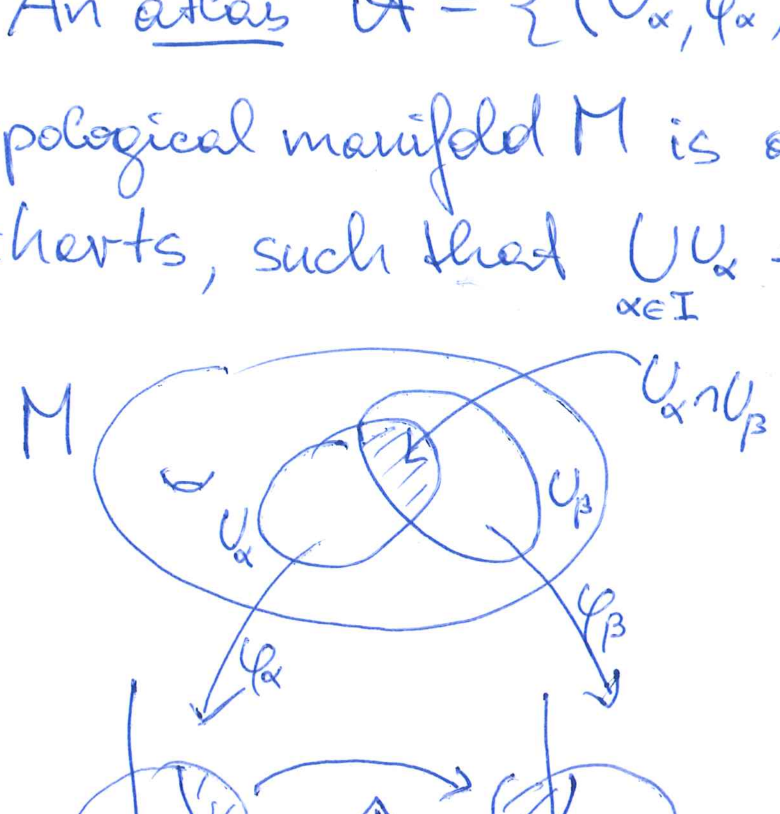
Smooth manifolds

<u>recall:</u> $\mathbb{R}^n \supset U \xrightarrow{f} V \subset \mathbb{R}^m$ is called <u>smooth</u> (or C^∞)
open
if it has continuous partial derivatives of all orders.

<u>Goal:</u> Want to make sense of smooth mfds & smooth functions between them.

<u>Idea:</u> define via coordinate charts.

<u>recall:</u> An <u>atlas</u> $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ for a topological manifold M is a collection of charts, such that $\bigcup_{\alpha\in I} U_\alpha = M$.



<u>Note:</u> transition functions $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are always continuous (since $\varphi_\alpha, \varphi_\beta$ are homeomorphisms)

Def: An atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ for a topological manifold is called smooth, if all the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are smooth.

Lemma: Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas $\overline{\mathcal{A}}$.

Proof: $\overline{\mathcal{A}}$ = set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. $\boxed{\Xi}$

Def: A smooth manifold is a topological manifold equipped with a $\underbrace{\text{maximal smooth atlas}}$.

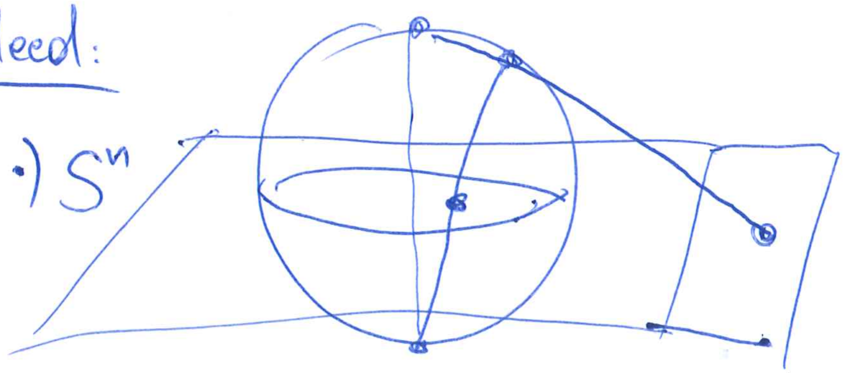
(also called "smooth structure")

Rmk: Similarly can define C^k, real-analytic, complex manifolds.

<u>Note:</u> All examples from last time are smooth mfds.

<u>Indeed:</u>

·) S^n



$$\varphi_N \circ \varphi_S^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, \quad \vec{x} \longmapsto \frac{\vec{x}}{|\vec{x}|^2}$$

·) products (eg T^n) ✓

·) projective space

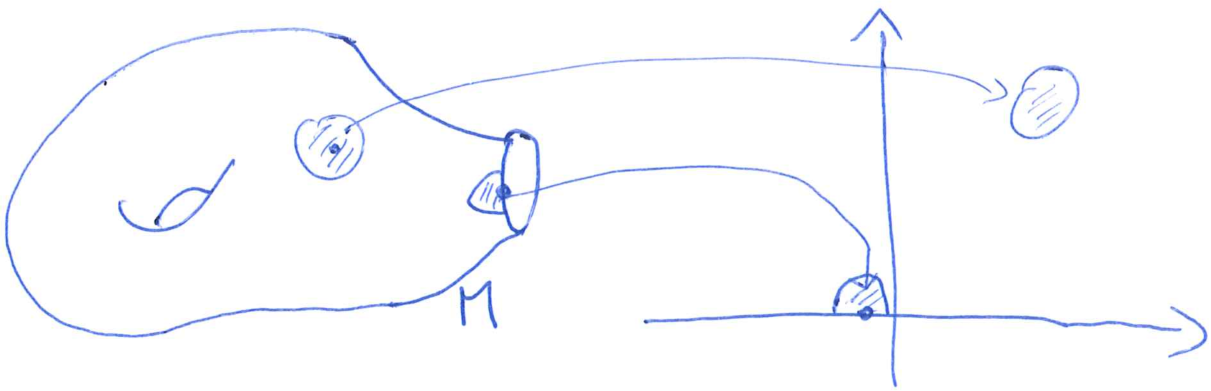
$$\varphi_1 \circ \varphi_0^{-1}(y_1, \dots, y_n) = \left(\frac{1}{y_1}, \frac{y_2}{y_1}, \dots, \frac{y_n}{y_1}\right)$$

smooth on $\mathbb{R}^n \setminus \{y_1 = 0\}$ as required, & similarly for all $\varphi_i \circ \varphi_j^{-1}$.

·) level sets of smooth fns & matrix groups:

use smooth implicit fn thm.

Manifolds with boundary



local model $\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$

Def: An n-dimensional topological manifold with boundary is a 2nd countable Hausdorff space that is locally modelled on $\mathbb{H}^n$.

$\partial \mathbb{H} = \{\vec{x} \in \mathbb{R}^n : x_n = 0\}$

$\partial M := \{p \in M \mid \varphi(p) \in \partial \mathbb{H}$ for some chart $(U, \varphi)\}$

$\mathrm{Int}\, M := M \setminus \partial M$

manifold boundary

($\neq$ topological boundary)

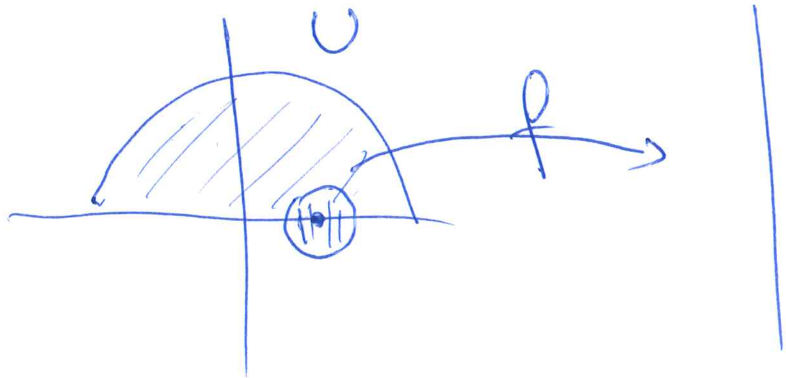
Note: ·) Int M is an n-dim top. mfd without boundary

·) ∂M is an $(n-1)$-dim top. mfd without boundary,

hence $\partial\partial M = \emptyset$.

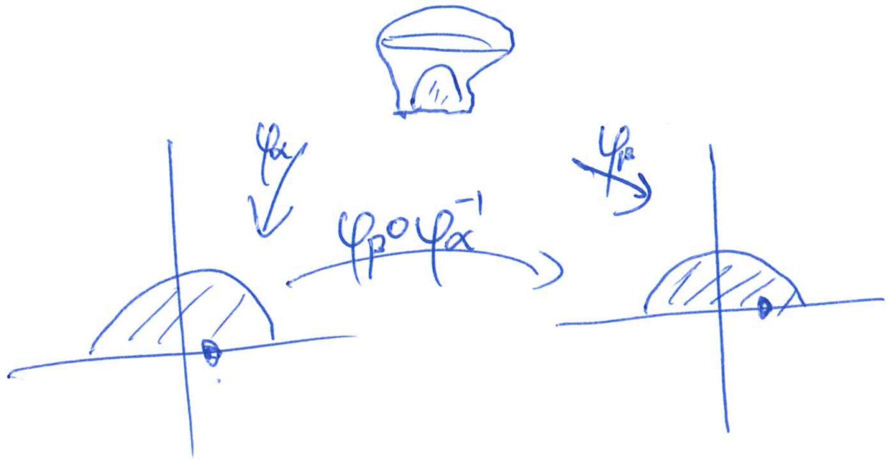
Rmk: Mfd $\equiv$ mfd with bdry whose bdry is empty.

recall: $\mathbb{H}^n \supset \underset{\text{open}}{U} \xrightarrow{f} \mathbb{R}^m$ smooth means $\forall p \in U \; \exists \tilde{U} \subset \mathbb{R}^n$ open & smooth extension $\tilde{f}: \tilde{U} \to \mathbb{R}^m$



eg on $\{x^2+y^2<1, y \geq 0\}$ the function $f(x,y) = y$ is smooth, but $f(x,y) = \sqrt{y}$ is not.

Def: A smooth manifold with boundary is a topological manifold with boundary with a smooth structure

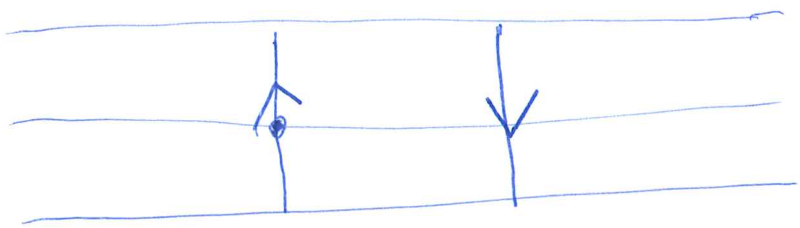


Note:

$$\varphi_\beta \circ \varphi_\alpha^{-1}(\partial \mathbb{H}) \subseteq \partial \mathbb{H}.$$

__Ex__ Möbius strip $M = \mathbb{R} \times [-1, 1] / \sim$

$$(x, y) \sim (x+1, -y)$$



__Note:__ $M \not\cong \underbrace{S^1 \times [-1, 1]}$

$\uparrow$ connected boundary

$\uparrow$ bdry has 2 components